

# Triangles

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## SAS Congruence Rule

### Congruency of Triangles

**Congruency** of triangles helps us find solutions to many problems in real life. For example, the distance travelled by a ball in a golf course is easy to measure when the ball is on land; however, when the land is separated by a water body like a pond or any other thing, the task of measurement becomes difficult. In such cases, we use the concept of congruency.

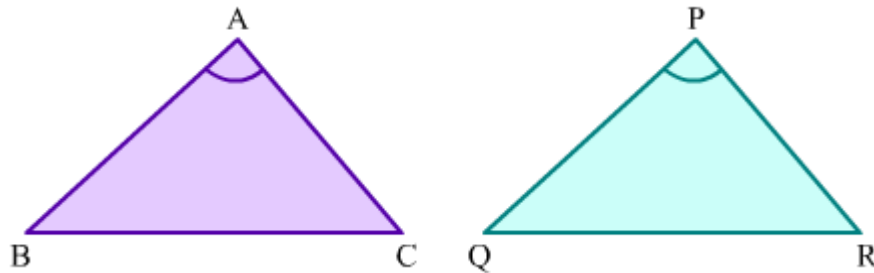
There are certain rules to check the congruency of triangles. One of these is the SAS (Side-Angle-Side) congruence rule. In this lesson, we will learn this rule and its applications.

### SAS Congruence Rule

Consider a triangle two of whose sides and the included angle are known. We can check for the congruency of this triangle with respect to another triangle if we know the corresponding sides and angle of that triangle. Two triangles can, thus, be termed 'congruent' or 'incongruent' by using the SAS congruence rule. This rule states that:

***If two sides of a triangle and the angle between them are equal to the corresponding sides and angle of another triangle, then the two triangles are congruent.***

Look at the given  $\triangle ABC$  and  $\triangle PQR$ .



Let us consider sides AB and AC and the included  $\angle BAC$  in  $\triangle ABC$ , and the corresponding sides and angle in  $\triangle PQR$ , i.e., PQ, PR and  $\angle QPR$ .

By the SAS congruence rule, the two triangles will be congruent if  $AB = PQ$ ,  $AC = PR$  and  $\angle BAC = \angle QPR$ .

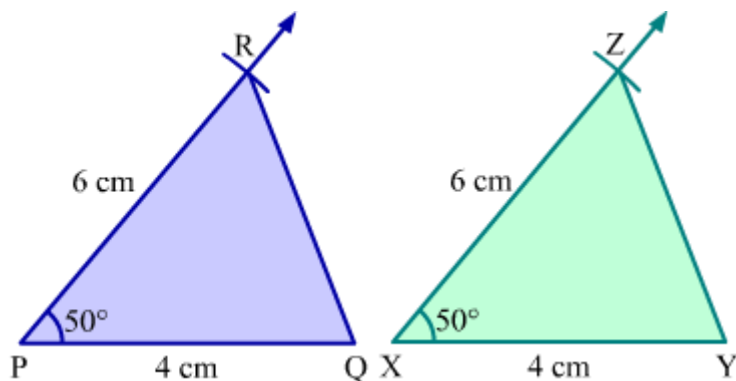
Similarly, we can check for congruency by taking other pairs of sides and included angles in these triangles.



## Verification of SAS Congruence Rule

The SAS congruence rule for triangles is taken as a postulate, so there is no proof for the same but we can verify it by doing an activity.

The steps of the activity are as follows:



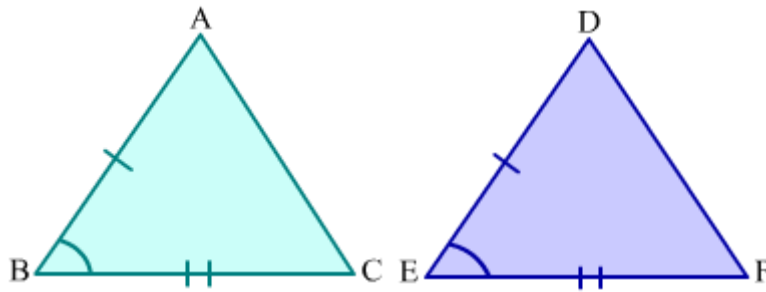
- i) Construct  $\Delta PQR$  such that  $PQ = 4$  cm,  $PR = 6$  cm and  $\angle QPR = 50^\circ$ .
- ii) Construct  $\Delta XYZ$  with the same measures such that  $XY = 4$  cm,  $XZ = 6$  cm and  $\angle YXZ = 50^\circ$ .
- iii) Cut both the triangles along their boundaries.
- iv) Try to superpose one triangle by the other. One triangle can be placed on the other in six different ways such that vertex lie on vertex.
- v) In one of the trials, you will get P falling over X, Q falling over Y and R falling over Z. In this case, you will see that both the triangles cover each other exactly.
- vi) Thus, under the correspondence  $PQR \leftrightarrow XYZ$ , the triangles are congruent.

This verifies the SAS congruence rule.

## CPCT

CPCT stands for 'corresponding parts of congruent triangles'. 'Corresponding parts' means corresponding sides and angles of triangles. According to CPCT, if two or more triangles are congruent to one another, then all of their corresponding parts are equal.

For example, in the given  $\Delta ABC$  and  $\Delta DEF$ ,  $AB = DE$ ,  $\angle B = \angle E$  and  $BC = EF$ . So, according to the SAS congruence criterion, we have  $\Delta ABC \cong \Delta DEF$ .



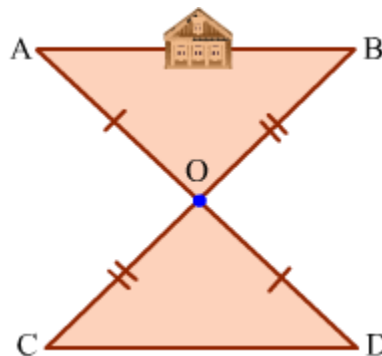
Now, by CPCT, we can say that the remaining corresponding parts of the two congruent triangles are also equal. This means that  $AC = DF$ ,  $\angle A = \angle D$  and  $\angle C = \angle F$ .

Similarly, we can apply CPCT in case of other congruent triangles.

### Solved Examples

#### Easy

**Example 1: Observe the following figure.**



Ajay wishes to determine the distance between two objects A and B, but there is a house in between. So, he devises an ingenious way to fix the problem. First, he fixes a pole at any point O so that both A and B are visible from O. He then fixes another pole at point D which is collinear to point O and object A, and is at the same distance from O as A, i.e.,  $DO = AO$ . Similarly, he fixes a pole at point C which is collinear to point O and object B, and is at the same distance from O as B, i.e.,  $CO = BO$ . Finally, he measures CD to find the distance between A and B. How can Ajay be sure that  $CD = AB$ ?

#### Solution:

We have two triangles in the given figure, i.e.,  $\triangle AOB$  and  $\triangle DOC$ .

In these two triangles, we have:

$$AO = DO \text{ (Given)}$$

$$\angle AOB = \angle DOC \text{ (Vertically opposite angles)}$$

$$BO = CO \text{ (Given)}$$

Therefore, by the SAS congruence rule, we can say that:

$$\triangle AOB \cong \triangle DOC$$

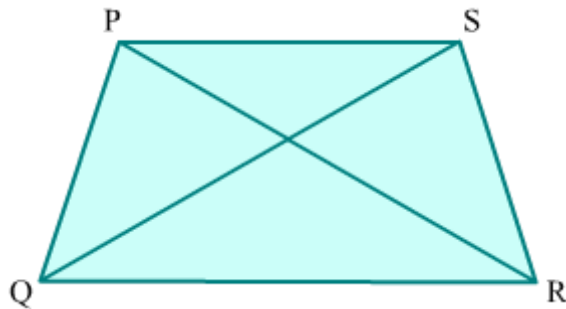
$$\Rightarrow AB = CD \text{ (By CPCT)}$$

This is the reason why Ajay measures CD to find the distance between objects A and B.

**Example 2:** In the given quadrilateral PQRS, PR bisects  $\angle QPS$  and  $PQ = PS$ . Prove that:

i)  $\triangle PQR \cong \triangle PSR$

ii)  $QR = SR$



**Solution:**

i) In  $\triangle PQR$  and  $\triangle PSR$ , we have:

$$PQ = PS \text{ (Given)}$$

$$PR = PR \text{ (Common side)}$$

$$\angle QPR = \angle SPR \text{ (because PR bisects } \angle QPS)$$

So, by the SAS congruence rule, we obtain:

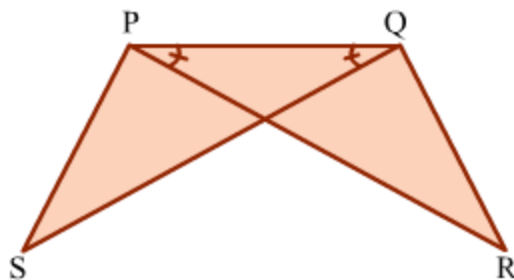
$$\triangle PQR \cong \triangle PSR$$

ii) We have proved that  $\triangle PQR \cong \triangle PSR$ .

$\therefore QR = SR$  (&because Corresponding parts of congruent triangles are equal)

### Medium

**Example 1:** In the shown figure,  $PR = QS$  and  $\angle QPR = \angle PQS$ . Prove that  $\triangle PQR \cong \triangle QPS$ . Also, show that  $PS = QR$  and  $\angle QPS = \angle PQR$ .



### Solution:

In  $\triangle PQR$  and  $\triangle QPS$ , we have:

$$PR = QS \text{ (Given)}$$

$$\angle QPR = \angle PQS \text{ (Given)}$$

$$PQ = PQ \text{ (Common side)}$$

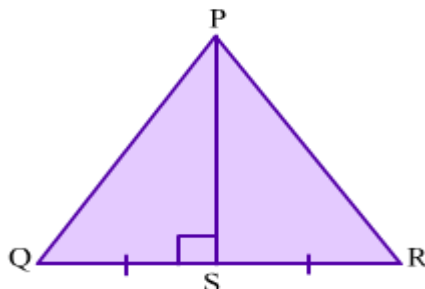
$$\therefore \triangle PQR \cong \triangle QPS \text{ (By the SAS congruence criterion)}$$

$$\Rightarrow PS = QR \text{ and } \angle QPS = \angle PQR \text{ (By CPCT)}$$

**Example 2:** Prove that  $\triangle PQR$  is isosceles if the altitude drawn from a vertex bisects the opposite side.

### Solution:

The given figure shows the  $\triangle PQR$  having  $PS$  as an altitude that bisects the opposite side  $QR$ .



In  $\triangle PSQ$  and  $\triangle PSR$ , we have:

$QS = SR$  (&because Altitude PS bisects QR)

$PS = PS$  (Common side)

$\angle PSQ = \angle PSR = 90^\circ$  (&because PS is the altitude to QR)

$\therefore \triangle PSQ \cong \triangle PSR$  (By the SAS congruence rule)

$\Rightarrow PQ = PR$  (By CPCT)

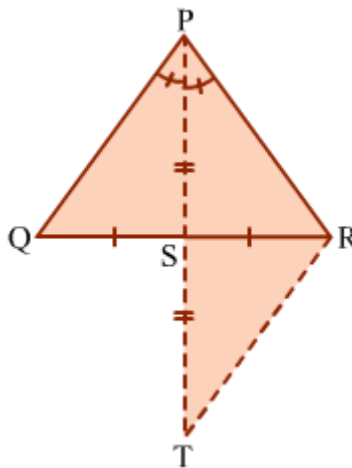
Therefore,  $\triangle PQR$  is isosceles.

**Example 3:** If the angle bisector of any angle of a triangle bisects the opposite side then show that the triangle is isosceles.

**Solution:**

Let  $\triangle PQR$  be the given triangle and PS is the angle bisector of  $\angle QPR$  such that it bisects the side QR.

Let us extend the segment PS to point T such that  $PS = TS$ .



In  $\triangle PQS$  and  $\triangle TRS$ , we have

$QS = RS$  (Given)

$\angle PSQ = \angle TSR$  (Vertically opposite angles)

$PS = TS$  (By construction)

So, by the SAS congruence criterion, we have:

$$\Delta PQS \cong \Delta TRS$$

By CPCT, we obtain

$$PQ = TR \quad \dots(1)$$

$$\text{And } \angle QPS = \angle RTS \quad \dots(2)$$

$$\text{But } \angle QPS = \angle RPS \quad \dots(3) \quad (\text{PS bisects } \angle QPR)$$

$$\therefore \angle RTS = \angle RPS \quad [\text{From (2) and (3)}]$$

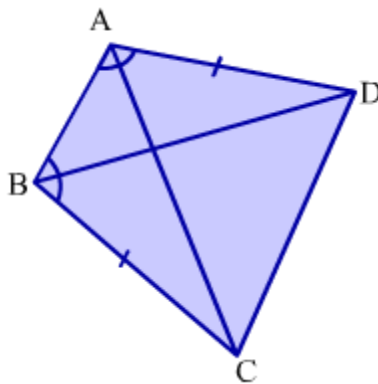
$$\Rightarrow PR = TR \quad (\text{Sides opposite to equal angles})$$

$$\therefore PQ = PR \quad [\text{From (1)}]$$

Thus,  $\Delta PQR$  is an isosceles triangle.

### Hard

**Example 1:** ABCD is a quadrilateral in which  $AD = BC$  and  $\angle DAB = \angle CBA$ .



Prove that:

i)  $\Delta ABD \cong \Delta BAC$

ii)  $BD = AC$

iii)  $\angle ABD = \angle BAC$

**Solution:**

i) In  $\triangle ABD$  and  $\triangle BAC$ , we have:

$$AD = BC \text{ (Given)}$$

$$\angle DAB = \angle CBA \text{ (Given)}$$

$$AB = BA \text{ (Common side)}$$

So, by the SAS congruence criterion, we have:

$$\triangle ABD \cong \triangle BAC$$

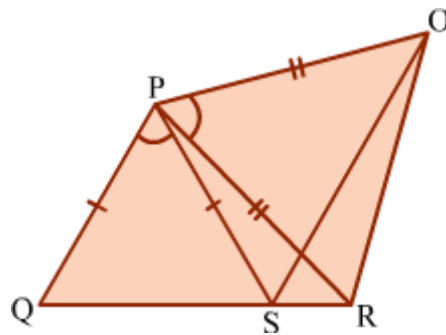
ii) We have proved that  $\triangle ABD \cong \triangle BAC$ .

$$\therefore BD = AC \text{ (By CPCT)}$$

iii) Since  $\triangle ABD \cong \triangle BAC$ , we have:

$$\angle ABD = \angle BAC \text{ (By CPCT)}$$

**Example 2:** In the given figure,  $PR = PO$ ,  $PQ = PS$  and  $\angle QPS = \angle OPR$ . Show that  $QR = SO$ .



**Solution:**

It is given that  $\angle QPS = \angle OPR$ .

$$\therefore \angle QPS + \angle SPR = \angle OPR + \angle SPR$$

$$\Rightarrow \angle QPR = \angle SPO \dots (1)$$

In  $\triangle QPR$  and  $\triangle SPO$ , we have:

$$PQ = PS \quad \text{(Given)}$$



$$\angle QPR = \angle SPO \quad (\text{From equation 1})$$

$$PR = PO \quad (\text{Given})$$

So, by the SAS congruence rule, we have:

$$\triangle QPR \cong \triangle SPO$$

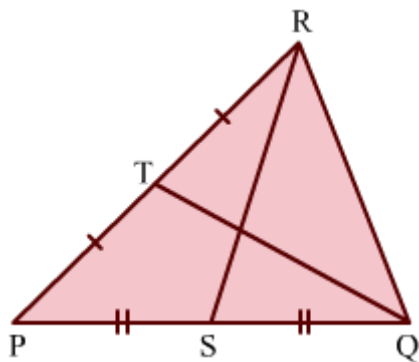
$$\Rightarrow QR = SO \quad (\text{By CPCT})$$

**Example 3:** In an isosceles triangle, prove that the medians on the equal sides are equal.

**Solution:**

Let  $\triangle PQR$  be an isosceles triangle such that  $PQ = PR$ .

Also, let  $RS$  and  $QT$  be the medians to the sides  $PQ$  and  $PR$  respectively.



In  $\triangle PQR$ , we have

$$PS = SQ = \frac{1}{2}PQ \quad (\text{RS is the median})$$

$$\text{And } PT = TR = \frac{1}{2}PR \quad (\text{QT is the median})$$

$$\text{But } PQ = PR$$

$$\therefore PS = SQ = PT = TR \quad \dots(1)$$

In  $\triangle PRS$  and  $\triangle PQT$ , we have

$$PQ = PR \quad (\text{Given})$$

$$\angle RPS = \angle QPT \quad (\text{Common angle})$$

$$PS = PT \quad [\text{From (1)}]$$

So, by the SAS congruence rule, we have:

$$\triangle PRS \cong \triangle PQT$$

$$\therefore RS = QT \quad (\text{By CPCT})$$

Thus, the medians on the equal sides of an isosceles triangle are equal.

Proving Theorem of Right Angled Triangle

There is a theorem of right angled triangles which states that:

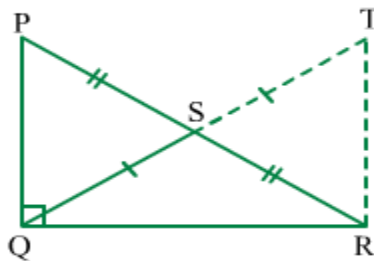
**In a right angled triangle, median drawn to the hypotenuse from the opposite vertex is equal to the half of the hypotenuse.**

Let us prove the theorem.

**Given:** Right angled  $\triangle PQR$ ,  $\angle PQR = 90^\circ$  and median  $QS$  to hypotenuse  $PR$ .

**To prove:**  $QS = \frac{1}{2}PR$

**Construction:** Extend  $QS$  to  $T$  such that  $QS = ST$  and join  $T$  to  $R$ .



**Proof:**

In  $\triangle PSQ$  and  $\triangle RST$ , we have

$$QS = TS \quad (\text{By construction})$$

$$\angle PSQ = \angle RST \quad (\text{Vertically opposite angles})$$

$$PS = RS \quad (\text{Given})$$

So, by SAS congruence criterion, we have

$$\triangle PSQ \cong \triangle RST$$

$$\therefore PQ = RT \quad \dots(1) \quad (\text{By CPCT})$$

$$\text{And } \angle QPS = \angle TRS \quad (\text{By CPCT})$$

Thus,  $PQ \parallel RT$  ( $\angle QPS$  and  $\angle TRS$  alternate interior angles formed by transversal PR)

Now, QR is also a transversal to parallel line segments PQ and RT.

$$\angle PQR + \angle TRQ = 180^\circ \quad (\text{Sum of interior angles on the same side of transversal})$$

$$\text{But } \angle PQR = 90^\circ$$

$$\therefore \angle TRQ = 90^\circ \quad (2)$$

In  $\triangle PQR$  and  $\triangle TRQ$ , we have

$$PQ = RT \quad [\text{From (1)}]$$

$$\angle PQR = \angle TRQ = 90^\circ$$

$$QR = QR \quad (\text{Common side})$$

So, by SAS congruence criterion, we have

$$\triangle PQR \cong \triangle TRQ$$

$$\therefore PR = QT \quad (\text{By CPCT})$$

$$\Rightarrow \frac{1}{2}PR = \frac{1}{2}QT$$

$$\text{But } \frac{1}{2}QT = QS$$

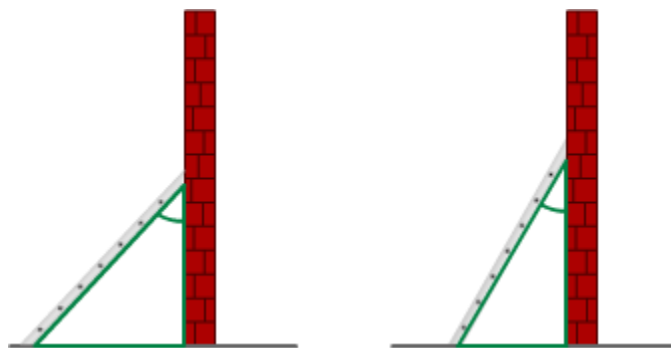
$$\therefore QS = \frac{1}{2}PR$$

Hence, proved.

### ASA Congruence Rule

### ASA Congruence Rule

Look at the given figure.



Observe how the ladder, the wall and the horizontal together make triangles in the figure. It can be seen that the angle marked between the ladder and the wall on the left is greater than the same angle marked on the right. Clearly, the triangles are not congruent although it is the same ladder on both sides.

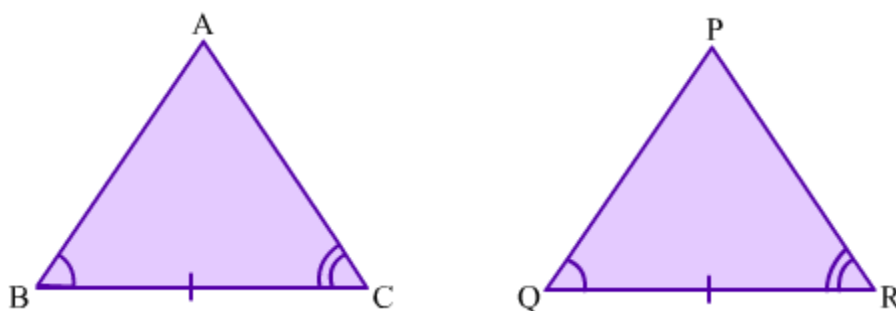
Both angles and sides play an important role in determining the congruency of triangles. In this lesson, we will discuss the ASA (Angle-Side-Angle) congruence rule and solve some problems based on it.

### ASA Congruence Rule

The ASA congruence rule for triangles states that:

***If two angles of a triangle and the side between them are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.***

Consider the given  $\triangle ABC$  and  $\triangle PQR$ .



Observe how corresponding components of the two triangles are marked.

Now, by the ASA congruence rule, the two triangles will be congruent if these corresponding components are equal, i.e., if  $\angle ABC = \angle PQR$ ,  $BC = QR$  and  $\angle ACB = \angle PRQ$ , then  $\triangle ABC \cong \triangle PQR$ .

Note that, under the above condition of congruence, we cannot write  $\triangle ABC \cong \triangle QRP$ . The order of the vertices matters in any congruency.

### Did You Know?

A bright meteor was seen in the sky above Greenland on December 9, 1997. In an attempt to find the fragments of the meteorite, scientists collected data from eyewitnesses who observed the meteor passing through the sky. As is shown in the figure below, the scientists considered sightlines of observers in different towns.

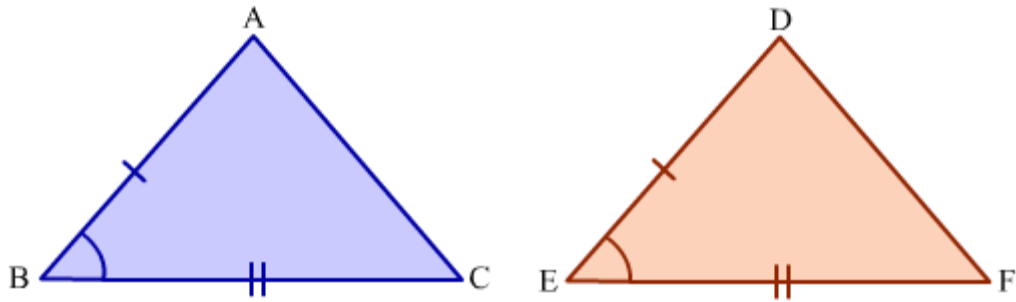


One such sightline was of observers in the town called Paamiut (Point P). Another was of observers in the town called Narsarsuaq (Point N). Using the ASA congruence rule, the scientists were able to gather enough information to successfully locate the fragments of the meteorite (Point M).

### Proof of the ASA Congruence Rule: Case 1

Let us consider  $\triangle ABC$  and  $\triangle DEF$  such that  $\angle ABC = \angle DEF$ ,  $\angle ACB = \angle DFE$  and  $BC = EF$ . By the ASA congruence rule,  $\triangle ABC$  and  $\triangle DEF$  are congruent. By CPCT, we have  $AB = DE$ .

**Case 1:** Let us prove  $\triangle ABC \cong \triangle DEF$  by taking  $AB = DE$ .



In this case, we have

$$AB = DE \text{ (Given)}$$

$$\angle ABC = \angle DEF \text{ (Given)}$$

$$BC = EF \text{ (Given)}$$

So, by the SAS congruence rule, we have:

$$\triangle ABC \cong \triangle DEF$$

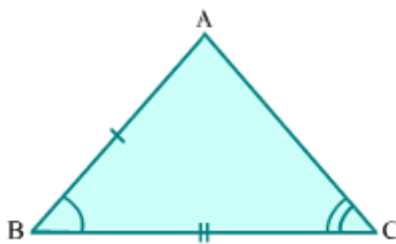
### Proof of the ASA Congruence Rule: Case 2

Let us consider  $\triangle ABC$  and  $\triangle DEF$  such that  $\angle ABC = \angle DEF$ ,  $\angle ACB = \angle DFE$  and  $BC = EF$ . By the ASA congruence rule,  $\triangle ABC$  and  $\triangle DEF$  are congruent. By CPCT, we have  $AB = DE$ . Let us assume  $AB \neq DE$ .

**Case 2:** Let us prove  $\triangle ABC \cong \triangle DEF$  by taking  $AB < DE$ .

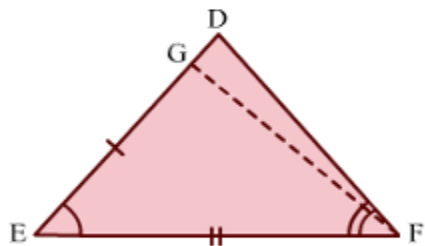
**Construction:** Mark a point G on DE such that  $GE = AB$ . Join G to F.

Now, in  $\triangle ABC$  and  $\triangle GEF$ , we have:



$$AB = GE \text{ (By construction)}$$

$$\angle ABC = \angle GEF \ (\because \angle ABC = \angle DEF \text{ and } \angle DEF = \angle GEF)$$



$$BC = EF \text{ (Given)}$$

So, by the SAS congruence rule, we obtain:

$$\triangle ABC \cong \triangle GEF$$

$$\Rightarrow \angle ACB = \angle GFE \text{ (By CPCT)}$$

$$\text{But } \angle ACB = \angle DFE \text{ (Given)}$$

$$\therefore \angle GFE = \angle DFE$$

This can be possible only when line segment GF coincides with line segment DF or point G coincides with point D. Therefore, AB must be equal to DE and  $\triangle GEF$  must be  $\triangle DEF$ .

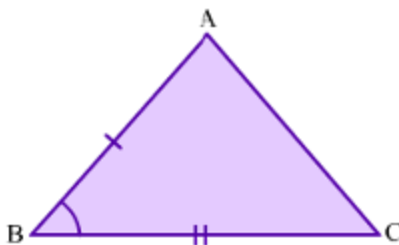
$$\therefore \triangle ABC \cong \triangle DEF$$

### Proof of the ASA Congruence Rule: Case 3

Let us consider  $\triangle ABC$  and  $\triangle DEF$  such that  $\angle ABC = \angle DEF$ ,  $\angle ACB = \angle DFE$  and  $BC = EF$ . By the ASA congruence rule,  $\triangle ABC$  and  $\triangle DEF$  are congruent. By CPCT, we have  $AB = DE$ . Let us assume  $AB \neq DE$ .

**Case 3:** Let us prove  $\triangle ABC \cong \triangle DEF$  by taking  $AB > DE$ .

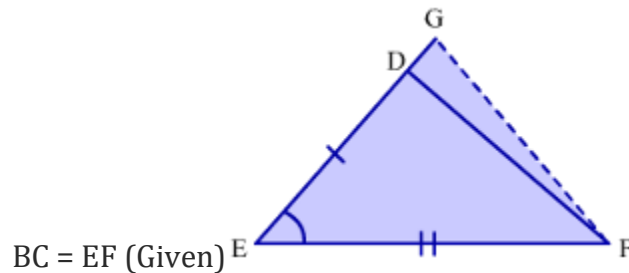
**Construction:** Extend ED to point G such that  $GE = AB$ . Join G to F.



Now, in  $\triangle ABC$  and  $\triangle GEF$ , we have:

$AB = GE$  (By construction)

$\angle ABC = \angle DEF$  ( $\because \angle ABC = \angle DEF$  and  $\angle DEF = \angle GEF$ )



So, by the SAS congruence rule, we obtain:

$\triangle ABC \cong \triangle GEF$

$\Rightarrow \angle ACB = \angle GFE$  (By CPCT)

But  $\angle ACB = \angle DFE$  (Given)

$\therefore \angle GFE = \angle DFE$

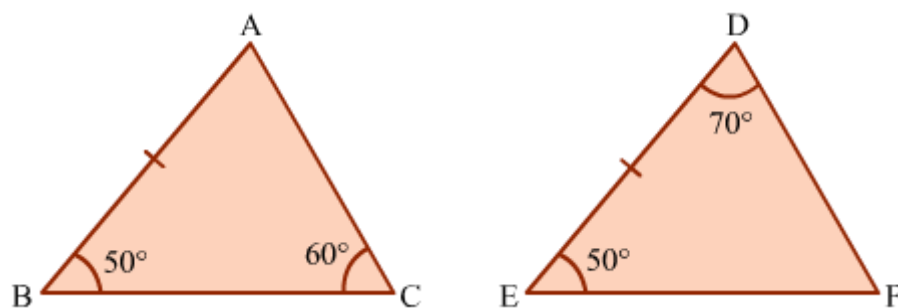
This can be possible only when line segment GF coincides with line segment DF or point G coincides with point D. Therefore, AB must be equal to DE and  $\triangle GEF$  must be  $\triangle DEF$ .

$\therefore \triangle ABC \cong \triangle DEF$

### Solved Examples

#### Easy

**Example 1:** Check whether the given triangles are congruent or not.





**Solution:**

In  $\triangle ABC$ , we have:

$$\angle ABC + \angle BCA + \angle BAC = 180^\circ \text{ (By the angle sum property)}$$

$$\Rightarrow 50^\circ + 60^\circ + \angle BAC = 180^\circ$$

$$\Rightarrow 110^\circ + \angle BAC = 180^\circ$$

$$\Rightarrow \angle BAC = 180^\circ - 110^\circ$$

$$\Rightarrow \angle BAC = 70^\circ$$

In  $\triangle ABC$  and  $\triangle DEF$ , we have:

$$\angle BAC = \angle EDF = 70^\circ$$

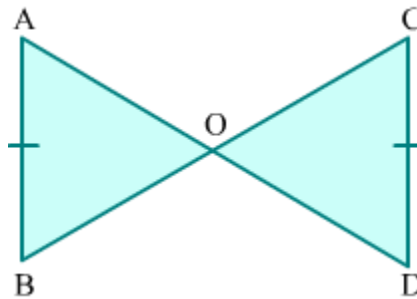
$$AB = DE \text{ (Given)}$$

$$\angle ABC = \angle DEF = 50^\circ$$

Therefore, by the ASA congruence rule, we have:

$$\triangle ABC \cong \triangle DEF$$

**Example 2:** In the given figure,  $AB$  and  $CD$  are two equal and parallel lines. Prove that  $\triangle ABO \cong \triangle CDO$ .

**Solution:**

It is given that  $AB \parallel CD$ .  $AD$  and  $BC$  are transversals lying on lines  $AB$  and  $CD$ .

So, by the alternate angles axiom, we obtain:



$$\angle OAB = \angle ODC \dots (1)$$

$$\angle OBA = \angle OCD \dots (2)$$

In  $\triangle ABO$  and  $\triangle CDO$ , we have:

$$\angle OAB = \angle ODC \text{ (By equation 1)}$$

$$AB = CD \text{ (Given)}$$

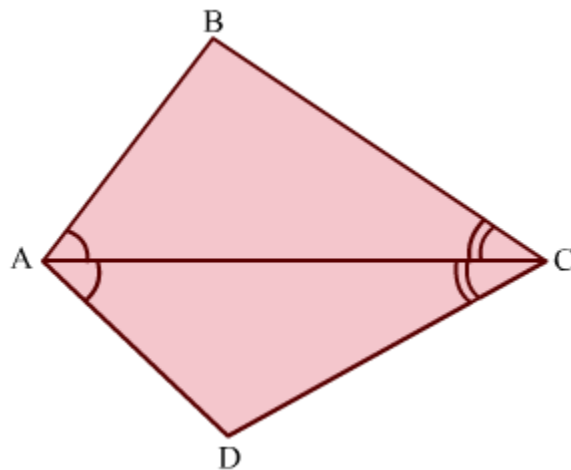
$$\angle OBA = \angle OCD \text{ (By equation 2)}$$

Thus, by the ASA congruence rule, we obtain:

$$\triangle ABO \cong \triangle CDO$$

### Medium

**Example 1:** In the given quadrilateral ABCD, diagonal AC bisects  $\angle BAD$  and  $\angle BCD$ . Prove that  
 $AB = AD$  and  $CB = CD$ .



### Solution:

Since diagonal AC bisects  $\angle BAD$  and  $\angle CAD$ , we have:

$$\angle BAC = \angle DAC \text{ and } \angle BCA = \angle DCA$$

In  $\triangle ACB$  and  $\triangle ACD$ , we have:

$$\angle BAC = \angle DAC \text{ (Given)}$$

$$\angle BCA = \angle DCA \text{ (Given)}$$

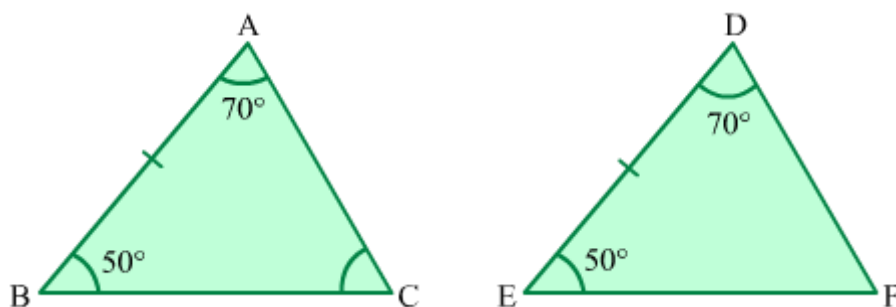
$$AC = AC \text{ (Common side)}$$

So, by the ASA congruence rule, we have:

$$\triangle ACB \cong \triangle ACD$$

$$\Rightarrow AB = AD \text{ and } CB = CD \text{ (By CPCT)}$$

**Example 2:** Consider the two triangular parks ABC and DEF shown below.



Tina jogs around park ABC and Aliya jogs around park DEF daily. Paths AB and DE are equal in length. If both girls jog an equal number of rounds daily, then check whether or not they cover the same distance while jogging?

**Solution:**

In  $\triangle ABC$  and  $\triangle DEF$ , we have:

$$\angle BAC = \angle EDF = 70^\circ \text{ (Given)}$$

$$AB = DE \text{ (Given)}$$

$$\angle ABC = \angle DEF = 50^\circ \text{ (Given)}$$

Therefore, by the ASA congruency rule, we obtain:

$$\triangle ABC \cong \triangle DEF$$

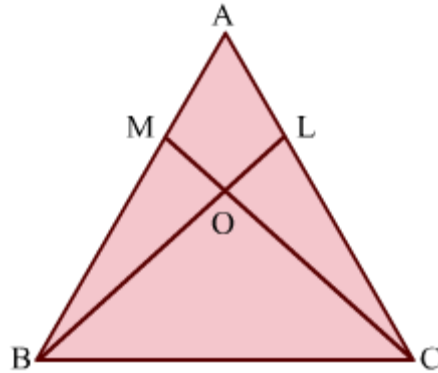
$$\Rightarrow AC = DF \text{ and } BC = EF \text{ (By CPCT)}$$

$$\therefore AB + BC + CA = DE + EF + FD$$

Hence, both Tina and Aliya cover the same distance daily while jogging.

## Hard

**Example 1:** The given  $\triangle ABC$  is isosceles with  $AB = AC$ .  $\angle LOC = 2\angle OBC$  and  $\angle MOB = 2\angle OCB$ . Prove that  $\triangle BCM \cong \triangle CBL$ .



### Solution:

It is given that:

$$\angle LOC = 2\angle OBC \dots (1)$$

$$\angle MOB = 2\angle OCB \dots (2)$$

Now,  $\angle LOC = \angle MOB$  (Vertically opposite angles)

Using equations (1) and (2), we obtain:

$$\angle OCB = \angle OBC$$

$$\Rightarrow \angle MCB = \angle LBC \dots (3)$$

In  $\triangle BCM$  and  $\triangle CBL$ , we have:

$$\angle MBC = \angle LCB \quad (\because \triangle ABC \text{ is isosceles with } AB = AC)$$

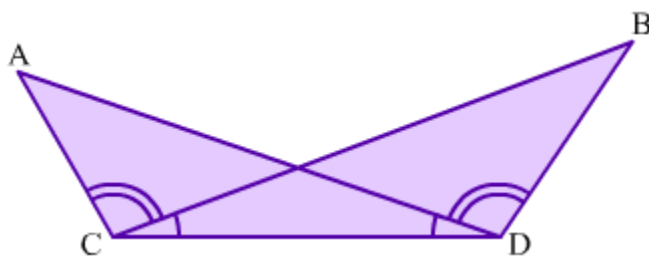
$$BC = CB \quad (\text{Common side})$$

$$\angle MCB = \angle LBC \quad (\text{Using equation 3})$$

Thus, by the ASA congruence rule, we obtain:

$$\triangle BCM \cong \triangle CBL$$

**Example 2:** In the given figure,  $\angle BCD = \angle ADC$  and  $\angle ACB = \angle BDA$ . Prove that  $AD = BC$  and  $\angle CAD = \angle DBC$ .



**Solution:**

It is given that:

$$\angle BCD = \angle ADC \dots (1)$$

$$\angle ACB = \angle BDA \dots (2)$$

On adding equations (1) and (2), we get:

$$\angle BCD + \angle ACB = \angle ADC + \angle BDA$$

$$\Rightarrow \angle ACD = \angle BDC \dots (3)$$

In  $\triangle ACD$  and  $\triangle BDC$ , we have:

$$\angle ADC = \angle BCD (\text{Given})$$

$$CD = DC \text{ (Common side)}$$

$$\angle ACD = \angle BDC \text{ (By equation 3)}$$

So, by the ASA congruence rule, we have:

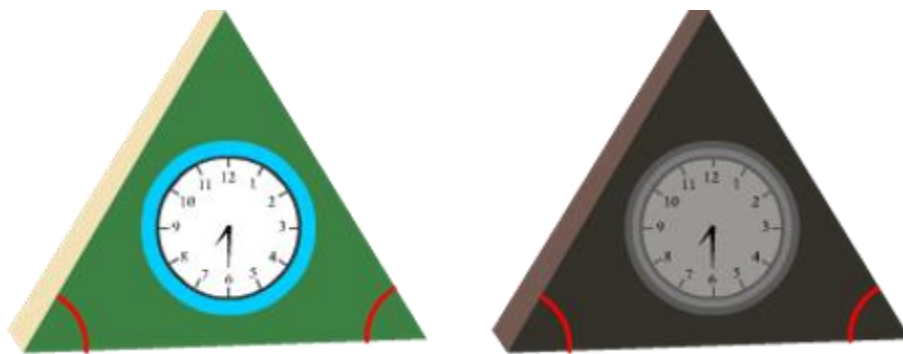
$$\triangle ACD \cong \triangle BDC$$

$$\Rightarrow AD = BC \text{ and } \angle CAD = \angle DBC \text{ (By CPCT)}$$

AAS Congruence Rule

**AAS Congruence Rule**

Consider the following triangular clocks.



In each clock, two angles on the base are marked. We can check the congruency of the clocks if we know the marked angles and the included side (i.e., the base) by using the ASA congruence rule.

Now, can we check the congruency of these clocks if instead of the base we are given any of the other two sides? Yes, we can do so and this is why we have the AAS (Angle-Angle-Side) congruence rule. In this lesson, we will discuss this rule and solve some examples related to it.

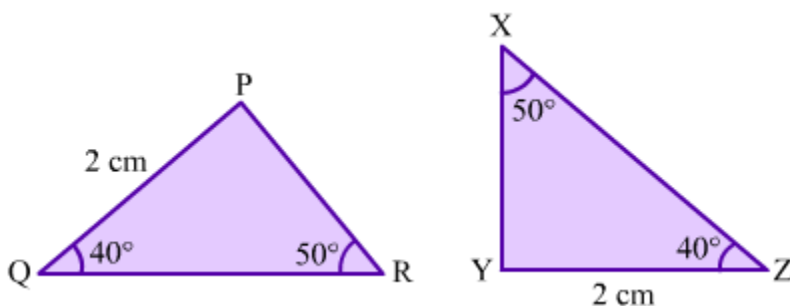
### AAS Congruence Rule

The AAS congruence rule for triangles states that:

***If any two angles and one side of a triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.***

This is nothing but the generalized form of the ASA congruence rule.

Consider the given  $\triangle PQR$  and  $\triangle XYZ$ .



In each triangle, the given side is not between the given angles. So, we cannot apply the ASA congruence rule here. Instead, we will use the AAS congruence rule.

In  $\triangle PQR$  and  $\triangle YZX$ , we have:

$$PQ = YZ = 2 \text{ cm}$$

$$\angle PQR = \angle YZX = 40^\circ$$

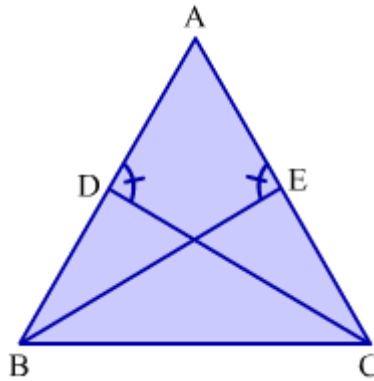
$$\angle PRQ = \angle YXZ = 50^\circ$$

Thus, by using the AAS congruence criterion, we obtain:  $\Delta PQR \cong \Delta YZX$

### Solved Examples

#### Easy

**Example 1:** The given  $\Delta ABC$  is isosceles with  $AB = AC$  and  $\angle ADC = \angle AEB$ . Prove that  $\Delta ABE \cong \Delta ACD$ .



#### Solution:

In  $\Delta ABE$  and  $\Delta ACD$ , we have:

$$AB = AC \text{ (Given)}$$

$$\angle AEB = \angle ADC \text{ (Given)}$$

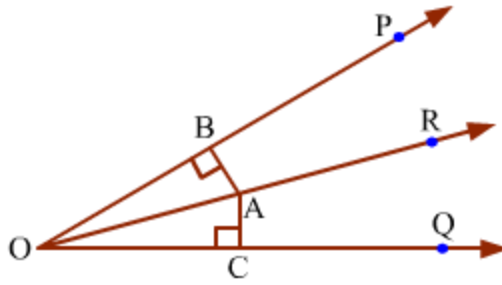
$$\angle BAE = \angle CAD \text{ (Common angle)}$$

Thus, by the AAS congruence rule, we obtain:

$$\Delta ABE \cong \Delta ACD$$

**Example 2:** In the given figure, OR bisects  $\angle POQ$  and A is any point on OR. AB and AC are the perpendiculars drawn from A to the arms OP and OQ respectively. Prove that  $\Delta AOB \cong \Delta AOC$ .





**Solution:**

It is given that OR bisects  $\angle POQ$ .

$$\therefore \angle POR = \angle QOR$$

$$\Rightarrow \angle BOA = \angle COA \dots (1)$$

In  $\triangle AOB$  and  $\triangle AOC$ , we have:

$$\angle ABO = \angle ACO = 90^\circ (\because AB \text{ and } AC \text{ are perpendiculars})$$

$$\angle BOA = \angle COA \text{ (By equation 1)}$$

$$AO = AO \text{ (Common side)}$$

Thus, by the AAS congruence rule, we obtain:

$$\triangle AOB \cong \triangle AOC$$

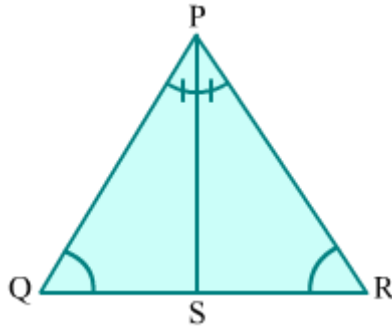
### Medium

**Example 1:** If two angles of a triangle are equal, then prove that the sides opposite them are also equal.

**Solution:**

Consider a  $\triangle PQR$  with  $\angle PQR = \angle PRQ$ .





We have to prove that  $PQ = PR$ .

**Construction:** Draw the bisector of  $\angle QPR$  and let it meet side  $QR$  at point  $S$ .

In  $\triangle PSQ$  and  $\triangle PSR$ , we have:

$$\angle PQS = \angle PRS \text{ (Given)}$$

$$PS = PS \text{ (Common side)}$$

$$\angle QPS = \angle RPS \text{ (}\because PS \text{ bisects } \angle QPR\text{)}$$

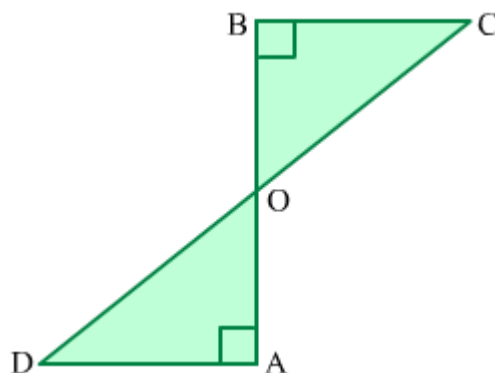
So, by the AAS congruence rule, we obtain:

$$\triangle PSQ \cong \triangle PSR$$

$$\Rightarrow PQ = PR \text{ (By CPCT)}$$

Hence, we have proved that the sides opposite equal angles of a triangle are also equal.

**Example 2:**  $DA$  and  $CB$  are equal perpendiculars to a line segment  $AB$ . Show that line segment  $CD$  bisects  $AB$  at point  $O$ .



**Solution:**

In  $\triangle DAO$  and  $\triangle CBO$ , we have:

$$\angle AOD = \angle BOC \text{ (Vertically opposite angles)}$$

$$\angle OAD = \angle OBC = 90^\circ \text{ (}\because \text{DA and CB are perpendiculars)}$$

$$DA = CB \text{ (Given)}$$

So, by the AAS congruence rule, we obtain:

$$\triangle DAO \cong \triangle CBO$$

$$\Rightarrow AO = BO \text{ (By CPCT)}$$

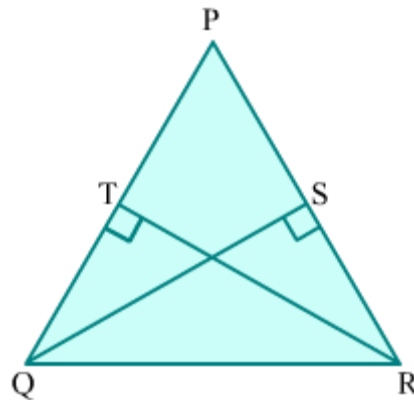
$$\text{Now, } AO + BO = AB$$

Thus, line segment CD bisects line segment AB at point O.

**Example 3:** If two altitudes of a triangle are equal then show that the triangle is isosceles.

**Solution:**

Let us consider  $\triangle PQR$  such that RT and QS are the equal altitudes drawn to the sides PQ and PR respectively.



In  $\triangle PTR$  and  $\triangle PSQ$ , we have

$$\angle PTR = \angle PSQ = 90^\circ$$

$$\angle TPR = \angle SPQ \text{ (Common angle)}$$



$$RT = QS \quad (\text{Given})$$

So, by the AAS congruence rule, we obtain:

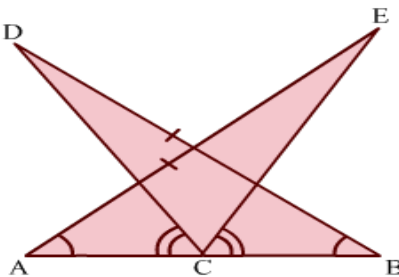
$$\Delta PTR \cong \Delta PSQ$$

$$\therefore PR = PQ \quad (\text{By CPCT})$$

Thus,  $\Delta PQR$  is an isosceles triangle.

### Hard

**Example 1:** In the given figure,  $\angle DBC = \angle EAC$ ,  $\angle DCA = \angle ECB$  and  $BD = AE$ . Prove that  $BC = AC$ .



### Solution:

It is given that  $\angle DCA = \angle ECB$ .

On adding  $\angle ECD$  to both sides of the above equation, we get:

$$\angle DCA + \angle ECD = \angle ECB + \angle ECD$$

$$\Rightarrow \angle ECA = \angle DCB \dots (1)$$

In  $\Delta BDC$  and  $\Delta AEC$ , we have:

$$\angle DCB = \angle ECA \quad (\text{From equation 1})$$

$$\angle DBC = \angle EAC \quad (\text{Given})$$

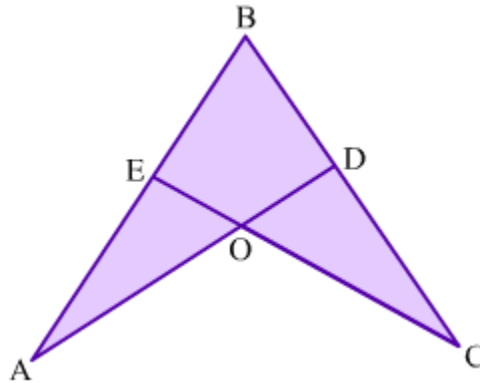
$$BD = AE \quad (\text{Given})$$

So, by the AAS congruence rule, we have:

$$\Delta BDC \cong \Delta AEC$$

$\Rightarrow BC = AC$  (By CPCT)

**Example 2:** In the figure,  $\angle BAD = \angle BCE$  and  $AB = CB$ . Prove that  $\triangle ABD \cong \triangle CBE$ .



**Solution:**

In  $\triangle AOE$  and  $\triangle COD$ , we have:

$$\angle EAO = \angle DCO \text{ (Given)}$$

$$\angle AOE = \angle COD \text{ (Vertically opposite angles)}$$

$$\therefore \angle AEO = \angle CDO \dots (1) \text{ [By the angle sum property]}$$

Now,

$$\angle AEO + \angle OEB = 180^\circ \text{ (Linear pair)}$$

$$\angle CDO + \angle ODB = 180^\circ \text{ (Linear pair)}$$

$$\therefore \angle AEO + \angle OEB = \angle CDO + \angle ODB$$

$$\Rightarrow \angle AEO + \angle OEB = \angle AEO + \angle ODB \text{ (Using equation (1))}$$

$$\Rightarrow \angle OEB = \angle ODB$$

$$\Rightarrow \angle CEB = \angle ADB \dots (2)$$

In  $\triangle ABD$  and  $\triangle CBE$ , we have:

$$\angle BAD = \angle BCE \text{ (Given)}$$

$$\angle ADB = \angle CEB \text{ (From equation 2)}$$

$$AB = CB \text{ (Given)}$$

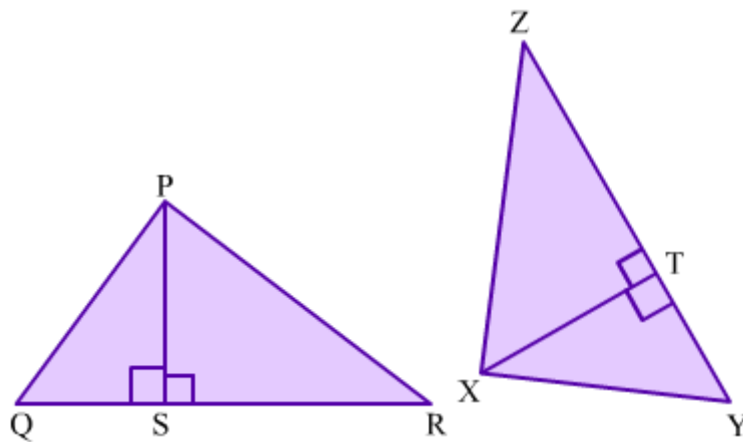
So, by the AAS congruence rule, we have:

$$\triangle ABD \cong \triangle CBE$$

**Example 3:** Prove that if two triangles are congruent then their corresponding altitudes are equal.

**Solution:**

Let PS and XT be corresponding altitudes of congruent triangles  $\triangle PQR$  and  $\triangle XYZ$ .



We have

$$\triangle PQR \cong \triangle XYZ$$

$$\therefore PQ = XY \quad \dots(1) \quad (\text{By CPCT})$$

$$\angle PQR = \angle XYZ \quad \dots(2) \quad (\text{By CPCT})$$

In  $\triangle PQS$  and  $\triangle XYT$ , we have

$$\angle PSQ = \angle XTY = 90^\circ$$

$$\angle PQR = \angle XYZ \quad [\text{From (2)}]$$

$$PQ = XY \quad [\text{From (1)}]$$

So, by the AAS congruence rule, we have:

$$\triangle PQS \cong \triangle XYT$$

$$\therefore PS = XT \quad (\text{By CPCT})$$

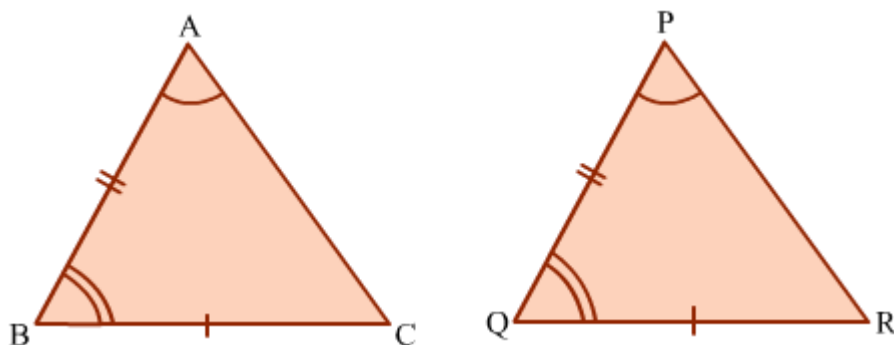
Thus, the corresponding altitudes of congruent triangles are equal.

### Difference between AAS and ASA

The AAS and ASA criteria are both used for checking or proving congruency of triangles, but there is a subtle difference between the two.

- The ASA rule is applied when two interior angles of a triangle and the side between them are considered.
- The AAS rule is applied when two interior angles of a triangle and any side other than the one between them are considered.

Take, for example, the given  $\triangle ABC$  and  $\triangle PQR$ .



To prove  $\triangle ABC \cong \triangle PQR$  by the ASA rule, we need:

$$\angle BAC = \angle QPR, \angle ABC = \angle PQR \text{ and } AB = PQ$$

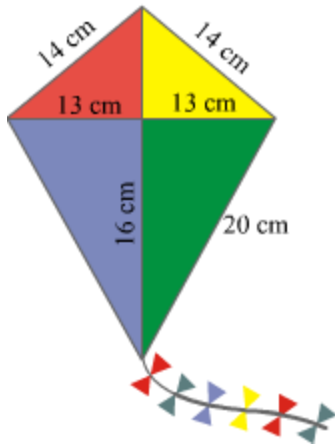
To prove  $\triangle ABC \cong \triangle PQR$  by the AAS rule, we need:

$$\angle BAC = \angle QPR, \angle ABC = \angle PQR \text{ and } BC = QR$$

SSS Congruence Rule

### Relation between the Congruency of Triangles and Their Sides

Consider the **kite** shown below.



It can be seen that the red and yellow coloured triangles have equal sides. On the basis of this information, can we say that the two triangles are congruent? Or, to rephrase the question, do the sides of triangles determine the congruency of the triangles? Yes, they do, and this is why we have the SSS (Side-Side-Side) congruence rule.

In this lesson, we will discuss the SSS congruence rule and its proof. We will also crack some problems based on it.

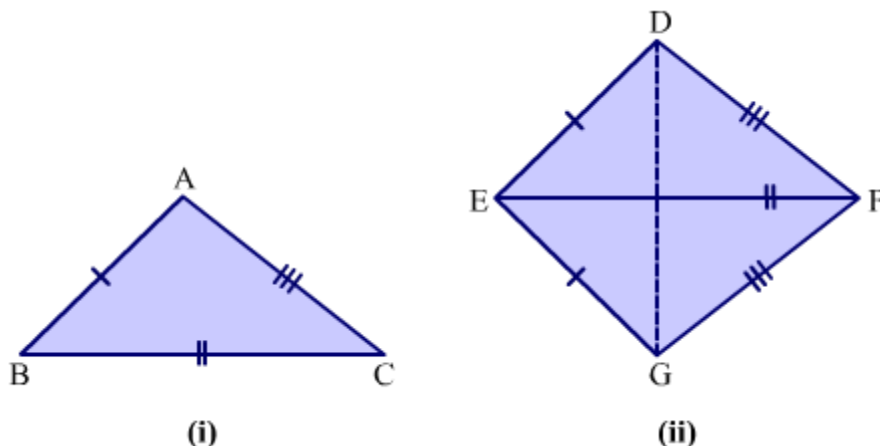
### SSS Congruence Rule

**Statement:** Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

**Given:**  $\triangle ABC$  and  $\triangle DEF$  such that  $AB = DE$ ,  $BC = EF$  and  $AC = DF$

**To prove:**  $\triangle ABC \cong \triangle DEF$

**Construction:** Suppose  $BC$  and  $EF$  are the longest sides of the two triangles. Draw  $EG$  such that  $\angle GEF = \angle ABC$  and  $GE = AB$ . Join point  $G$  to points  $F$  and  $D$ .



**Proof:** In  $\triangle ABC$  and  $\triangle GEF$ , we have:

$$BC = EF \text{ (Given)}$$

$$AB = GE \text{ (By construction)}$$

$$\angle ABC = \angle GEF \text{ (By construction)}$$

So, by the SAS congruence rule, we have:

$$\triangle ABC \cong \triangle GEF$$

$$\Rightarrow \angle BAC = \angle EGF \text{ and } AC = GF \text{ (By CPCT)}$$

$$\text{Now, } AB = DE \text{ and } AB = GE$$

$$\Rightarrow DE = GE \dots (1)$$

$$\text{Similarly, } AC = DF \text{ and } AC = GF$$

$$\Rightarrow DF = GF \dots (2)$$

In  $\triangle DEG$ , we have:

$$DE = GE \text{ (From equation 1)}$$

$$\Rightarrow \angle EDG = \angle EGD \dots (3)$$

In  $\triangle DFG$ , we have:

$$DF = GF \text{ (From equation 2)}$$

$$\Rightarrow \angle FDG = \angle FGD \dots (4)$$

On adding equations 3 and 4, we get:

$$\angle EDG + \angle FDG = \angle EGD + \angle FGD$$

$$\Rightarrow \angle EDF = \angle EGF$$

We know that  $\angle EGF = \angle BAC$  (Proved above)

$$\therefore \angle BAC = \angle EDF \dots (5)$$



Thus, in  $\triangle ABC$  and  $\triangle DEF$ , we have:

$$AB = DE \text{ (Given)}$$

$$\angle BAC = \angle EDF \text{ (From equation 5)}$$

$$AC = DF \text{ (Given)}$$

So, by the SAS congruence rule, we obtain:

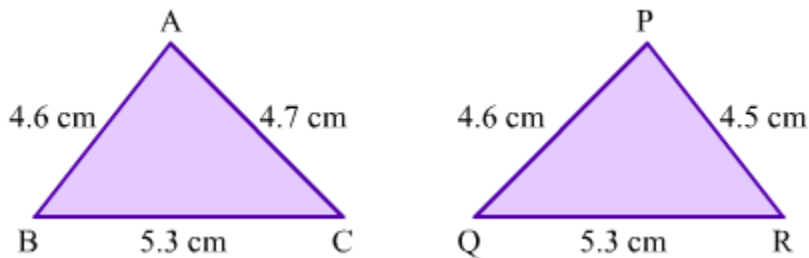
$$\triangle ABC \cong \triangle DEF$$

Hence, the SSS congruence rule holds true.

### Solved Examples

#### Easy

**Example 1:** Are the following triangles congruent?



**Solution:**

In  $\triangle ABC$  and  $\triangle PQR$ , we have:

$$AB = PQ = 4.6 \text{ cm}$$

$$BC = QR = 5.3 \text{ cm}$$

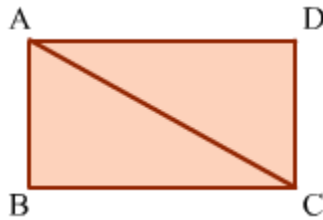
$$\text{But } AC \neq PR$$

Therefore,  $\triangle ABC$  and  $\triangle PQR$  are not congruent.

**Example 2:** ABCD is a rectangle with AC as one of its diagonals. Prove that the triangles formed on the two sides of diagonal AC are congruent.

**Solution:**

The required rectangle ABCD with AC as its diagonal can be drawn as is shown.



In  $\triangle ABC$  and  $\triangle CDA$ , we have:

$AB = CD$  ( $\because$  Opposite sides of a rectangle are equal)

$BC = DA$  ( $\because$  Opposite sides of a rectangle are equal)

$CA = AC$  (Common side)

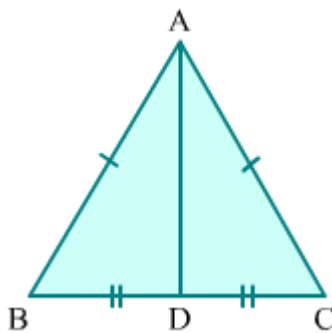
Therefore, by the SSS congruence rule, we have:

$\triangle ABC \cong \triangle CDA$

Thus, the triangles formed on the two sides of diagonal AC are congruent.

### Medium

**Example 1:** The given  $\triangle ABC$  is isosceles with  $AB = AC$ . AD is a median of the triangle. Prove that AD is perpendicular to BC.



### Solution:

In  $\triangle ABD$  and  $\triangle ACD$ , we have:

$AB = AC$  (Given)

$BD = DC$  ( $\because$  D is the midpoint of BC)

$AD = AD$  (Common side)

Therefore, by the SSS congruence rule, we obtain:

$$\triangle ABD \cong \triangle ACD$$

$$\Rightarrow \angle ADB = \angle ADC \text{ (By CPCT)}$$

Also,  $\angle ADB$  and  $\angle ADC$  form a linear pair.

$$\text{So, } \angle ADB + \angle ADC = 180^\circ$$

$$\Rightarrow \angle ADB + \angle ADB = 180^\circ \text{ } (\because \angle ADB = \angle ADC)$$

$$\Rightarrow 2\angle ADB = 180^\circ$$

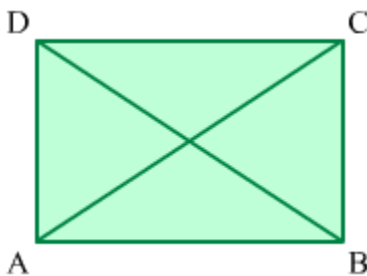
$$\Rightarrow \angle ADB = 90^\circ$$

Thus,  $\angle ADB = \angle ADC = 90^\circ$ , which means that AD is perpendicular to BC.

**Example 2:** ABCD is a parallelogram. If the diagonals of ABCD are equal, then find the measure of  $\angle ABC$ .

**Solution:**

The given parallelogram ABCD with equal diagonals AC and BD is shown below.



In parallelogram ABCD, we have:

$$AB = CD \text{ and } AD = BC \text{ } (\because \text{Opposite sides of a parallelogram are equal})$$

In  $\triangle ADB$  and  $\triangle BCA$ , we have:

$$AD = BC \text{ (Proved above)}$$

$$BD = AC \text{ (Given)}$$

$$BA = AB \text{ (Common side)}$$

So, by the SSS congruence rule, we have:

$$\triangle ADB \cong \triangle BCA$$

$$\Rightarrow \angle BAD = \angle ABC \dots (1) \text{ [By CPCT]}$$

Now, AD is parallel to BC and the transversal AB intersects them at A and B respectively.

We know that the sum of the interior angles on the same side of a transversal is supplementary.

$$\therefore \angle BAD + \angle ABC = 180^\circ$$

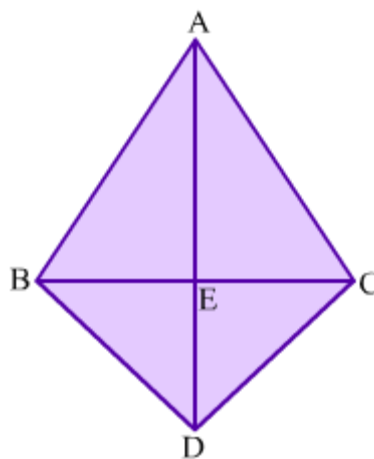
$$\Rightarrow \angle ABC + \angle ABC = 180^\circ \text{ (By equation 1)}$$

$$\Rightarrow 2\angle ABC = 180^\circ$$

$$\Rightarrow \angle ABC = 90^\circ$$

### Hard

**Example 1:** In the given figure,  $\triangle ABC$  and  $\triangle DBC$  are isosceles with  $AB = AC$  and  $DB = DC$ . Prove that AD is the perpendicular bisector of BC.



### Solution:

In  $\triangle ABD$  and  $\triangle ACD$ , we have:

$$AB = AC \text{ (Given)}$$

$$DB = DC \text{ (Given)}$$

$$AD = AD \text{ (Common side)}$$

So, by the SSS congruence rule, we have:

$$\triangle ABD \cong \triangle ACD$$

$$\Rightarrow \angle BAE = \angle CAE \dots (1) \text{ [By CPCT]}$$

In  $\triangle BAE$  and  $\triangle CAE$ , we have:

$$AB = AC \text{ (Given)}$$

$$\angle BAE = \angle CAE \text{ (From equation 1)}$$

$$AE = AE \text{ (Common side)}$$

So, by the SAS congruence rule, we have:

$$\triangle BAE \cong \triangle CAE$$

$$\Rightarrow BE = CE \text{ and } \angle BEA = \angle CEA \text{ (By CPCT)}$$

We know that  $\angle BEA + \angle CEA = 180^\circ$  as they form a linear pair.

$$\text{So, } 2\angle BEA = 180^\circ \text{ (Proved above that } \angle BEA = \angle CEA)$$

$$\Rightarrow \angle BEA = 90^\circ$$

$$\text{Therefore, } \angle BEA = \angle CEA = 90^\circ$$

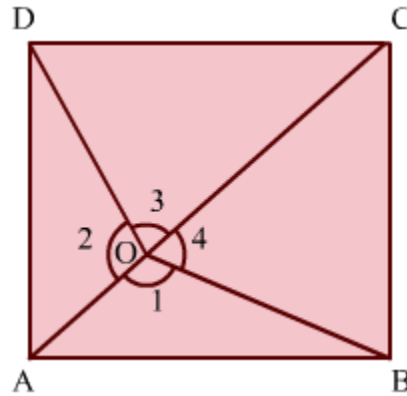
Since  $BE = CE$  and  $\angle BEA = \angle CEA = 90^\circ$ , AD is the perpendicular bisector of BC.

**Example 2:** O is a point inside a square ABCD such that it is at an equal distance from points B and D. Prove that points A, O and C are collinear.

**Solution:**

The square with the given specifications is drawn as is shown.

**Construction:** Join point O to the vertices of the square.



In  $\triangle AOD$  and  $\triangle AOB$ , we have:

$AD = AB$  (Sides of a square)

$AO = AO$  (Common side)

$OD = OB$  (Given)

So, by the SSS congruence rule, we have:

$\triangle AOD \cong \triangle AOB$

$\Rightarrow \angle 1 = \angle 2 \dots (1)$  [By CPCT]

Similarly,  $\triangle DOC \cong \triangle BOC$

$\Rightarrow \angle 3 = \angle 4 \dots (2)$  [By CPCT]

We know that:

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360^\circ$$

$$\Rightarrow 2\angle 2 + 2\angle 3 = 360^\circ \text{ (From equations 1 and 2)}$$

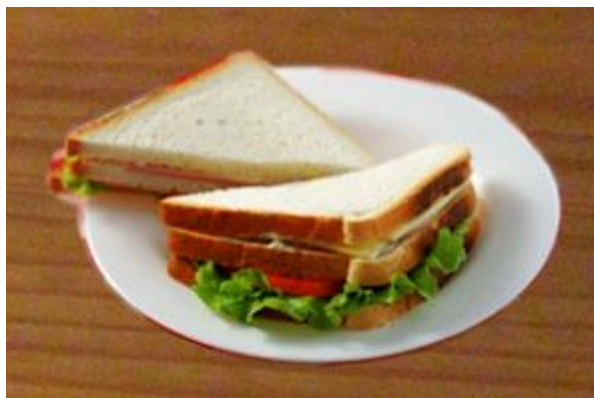
$$\Rightarrow \angle 2 + \angle 3 = 180^\circ$$

Thus,  $\angle 2$  and  $\angle 3$  form a linear pair. Therefore, AOC is a line; in other words, points A, O and C are collinear.

**RHS Congruence Rule**

**RHS Congruence Rule**

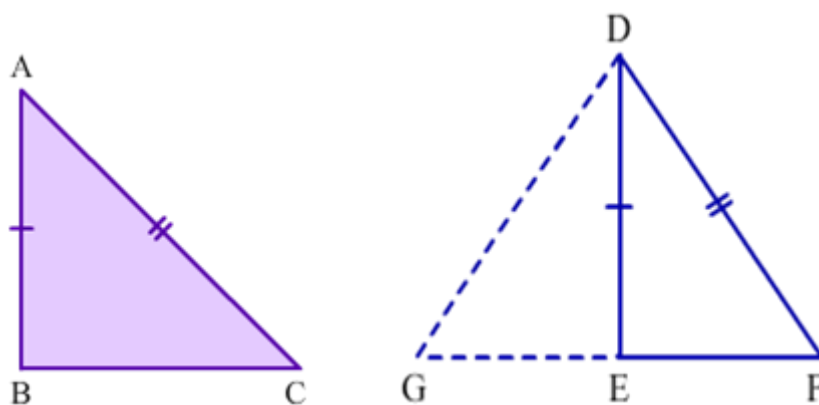
Right angles are all around us. For example, while building houses, the walls are kept at a right angle to the horizontal. Different square and rectangular figures surround us and all of them consist of right angles. The following figure also shows right angles.



In the figure, the pieces of bread resemble right-angled triangles. They also seem to be congruent. Right-angled triangles are special and their congruency is checked by a special congruence rule known as the RHS (Right-Hypotenuse-Side) rule.

We will study the RHS congruence rule in this lesson and solve some examples to familiarize ourselves with the concept.

**RHS congruence theorem:** Two right-angled triangles are congruent if the hypotenuse and a side of one triangle are equal to the hypotenuse and the corresponding side of the other triangle.



**Given:** Two right-angled triangle ABC and DEF such that  $\angle B = \angle E = 90^\circ$ ; Hypotenuse AC = Hypotenuse DF and  $AB = DE$ .

**To prove:**  $\triangle ABC \cong \triangle DEF$ .

**Construction:** Produce FE to G so that  $EG = BC$  and join DG.



**Proof:**

In triangles ABC and DEF,

$$AB = DE \quad (\text{Given})$$

$$BC = EG \quad (\text{By construction})$$

$$\angle ABC = \angle DEF \quad (\text{Each equal to } 90^\circ)$$

Thus, by SAS congruence criterion,

$$\triangle ABC \cong \triangle DEG$$

$$\Rightarrow \angle ACB = \angle DGE \text{ and } AC = DG \quad (\text{CPCT})$$

$$\text{Given, } AC = DF$$

$$\therefore DG = AC = DF$$

In  $\triangle DGF$ , we have

$$DG = DF$$

$$\angle G = \angle F \quad (\text{Angles opposite to equal sides are equal})$$

In  $\triangle DEF$  and  $\triangle DEG$ ,

$$\angle G = \angle F \quad (\text{Proved})$$

$$\angle DEG = \angle DEF \quad (\text{Both equal to } 90^\circ)$$

$$\text{Thus, } \angle GDE = 180^\circ - (\angle G + \angle DEG) = 180^\circ - (\angle F + \angle DEF) = \angle FDE$$

In  $\triangle DEG$  and  $\triangle DEF$ ,

$$DG = DF \quad (\text{Proved})$$

$$DE = DE \quad (\text{Common})$$

$$\angle GDE = \angle FDE \quad (\text{Proved})$$

Thus, by SAS congruence criterion

$$\triangle DEG \cong \triangle DEF$$

$$\text{But, we have } \triangle ABC \cong \triangle DEG$$

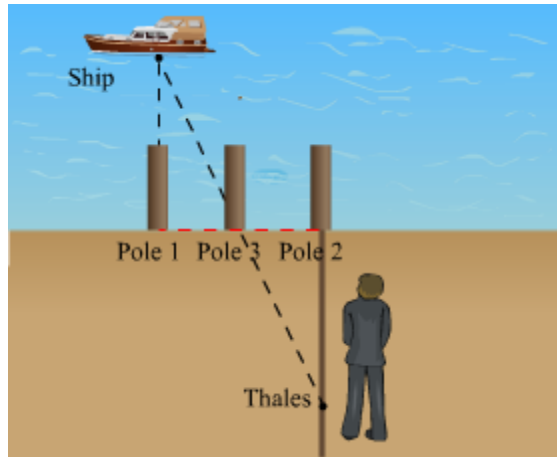
$$\text{Thus, } \triangle ABC \cong \triangle DEF$$

**Whiz Kid**

RHS congruence rule was used by the famous ancient Greek mathematician Thales to calculate the distance of a ship anchored at sea from the shore. For this, he stuck three poles on the shore such that the first one was directly in front of the ship, the second was at some distance from the first pole and the third was exactly between the other two poles. He then walked backward along a line from the second pole perpendicular to the shore until the middle pole and the ship were in the same line of sight. Then, he marked his position. This is shown in the following figure.







It can be seen that the triangle formed on the sea is congruent to the triangle formed on the shore by the RHS rule. So, the distance between the ship and the shore is equal to the distance between the second pole and the spot where Thales stands.

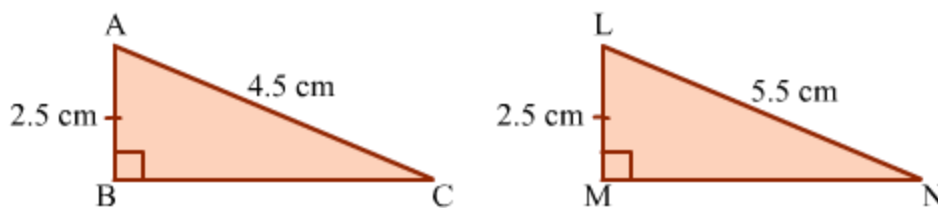
### Solved Examples

#### Easy

**Example 1:**  $\triangle ABC$  and  $\triangle LMN$  are right-angled at  $\angle ABC$  and  $\angle LMN$  respectively. In  $\triangle ABC$ ,  $AB = 2.5$  cm and  $AC = 4.5$  cm. In  $\triangle LMN$ ,  $LN = 5.5$  cm and  $LM = 2.5$  cm. Examine whether the two triangles are congruent.

#### Solution:

On the basis of the given information, the two triangles can be drawn as is shown.



In  $\triangle ABC$  and  $\triangle LMN$ , we have:

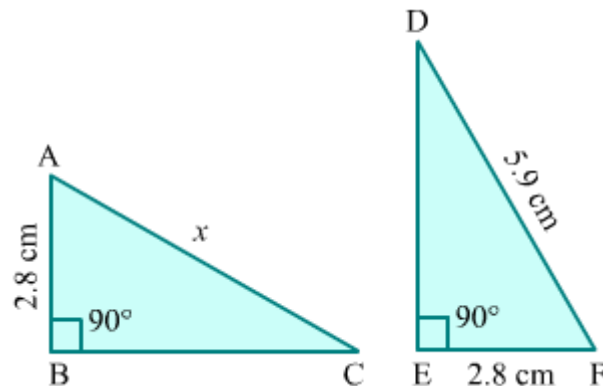
$\angle ABC = \angle LMN$  (Right angles)

$AB = LM = 2.5$  cm (Given)

But  $AC \neq LN$

Hence,  $\triangle ABC$  and  $\triangle LMN$  are not congruent.

**Example 2:** Find the value of  $x$  if the shown triangles ABC and DEF are congruent.



**Solution:**

It is given that  $\triangle ABC \cong \triangle DEF$ .

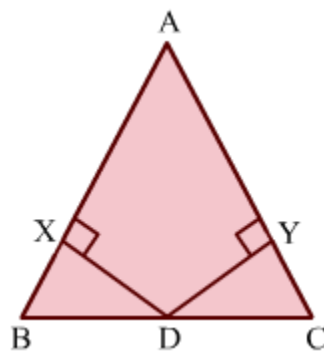
When two triangles are congruent, their corresponding sides are equal.

$$\therefore AC = DF = 5.9\text{ cm}$$

Thus, the value of  $x$  is  $5.9\text{ cm}$ .

**Medium**

**Example 1:** In the given  $\triangle ABC$ ,  $D$  is the midpoint of side  $BC$ . The perpendiculars  $DX$  and  $DY$  drawn from point  $D$  to sides  $AB$  and  $BC$  respectively are of the same length. Prove that  $DX$  and  $DY$  make the same angle with  $BC$ .



**Solution:**

On comparing  $\triangle DXB$  and  $\triangle DYC$ , we get:

$$DX = DY \text{ (Given)}$$

$$\angle DXB = \angle DYC = 90^\circ \text{ (}\because \text{DX and DY are perpendiculars)}$$

$$BD = CD \text{ (}\because \text{D is the midpoint of BC)}$$

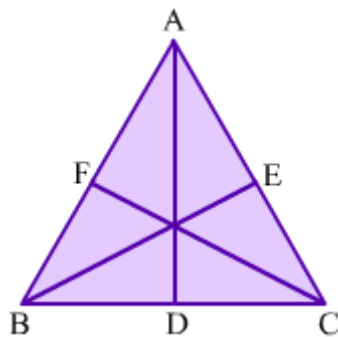
Thus, by the RHS congruence rule, we have:

$$\triangle DXB \cong \triangle DYC$$

$$\Rightarrow \angle BDX = \angle CDY \text{ (By CPCT)}$$

Thus, the perpendiculars DX and DY make the same angle with side BC.

**Example 2:** In the given  $\triangle ABC$ , AD, BE and CF are the altitudes. If the three altitudes are equal, then prove that the triangle is equilateral.



**Solution:**

In  $\triangle BEC$  and  $\triangle CFB$ , we have:

$$BC = CB \text{ (Common side)}$$

$$BE = CF \text{ (Given)}$$

$$\angle BEC = \angle CFB = 90^\circ \text{ (}\because \text{BE and CF are altitudes)}$$

So, by the RHS congruence rule, we obtain:

$$\triangle BEC \cong \triangle CFB$$

$$\Rightarrow \angle BCE = \angle CBF \text{ (By CPCT)}$$

$$\Rightarrow \angle CBA = \angle BCA$$

$\Rightarrow AC = AB \dots (1) [\because \text{Sides opposite equal angles are equal}]$

Similarly, we can prove that  $\triangle ADB \cong \triangle BEA$ .

$\Rightarrow \angle DBA = \angle BAE$  (By CPCT)

$\Rightarrow \angle CBA = \angle BAC$

$\Rightarrow AC = BC \dots (2) [\because \text{Sides opposite equal angles are equal}]$

From equations (1) and (2), we get:

$$AB = BC = AC$$

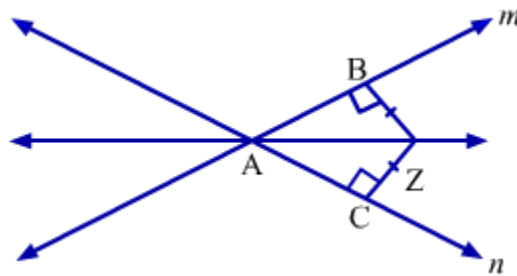
Hence,  $\triangle ABC$  is equilateral.

### Hard

**Example 1:** If  $Z$  is a point equidistant from two lines  $m$  and  $n$  intersecting at point  $A$ , then prove that  $AZ$  bisects the angle between  $m$  and  $n$ .

#### Solution:

The following figure can be drawn as per the given information.



**Construction:**  $ZB$  and  $ZC$  are perpendiculars drawn from point  $Z$  to lines  $m$  and  $n$  respectively.

Since  $Z$  is equidistant from  $m$  and  $n$ , we have:

$$ZB = ZC$$

In  $\triangle ZBA$  and  $\triangle ZCA$ , we have:

$$ZB = ZC \text{ (Proved above)}$$

$$\angle ZBA = \angle ZCA = 90^\circ \ (\because ZB \text{ and } ZC \text{ are perpendiculars})$$

$$ZA = ZA \text{ (Common side)}$$

So, by the RHS congruence rule, we have:

$$\Delta ZBA \cong \Delta ZCA$$

$$\Rightarrow \angle ZAB = \angle ZAC \text{ (By CPCT)}$$

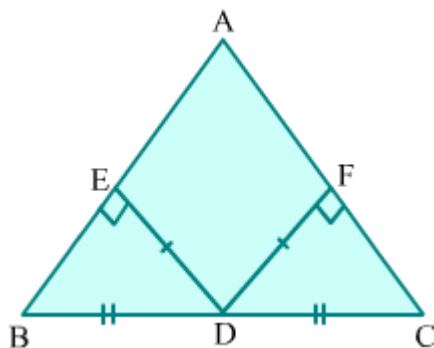
$$\text{Now, } \angle ZAB + \angle ZAC = \angle BAC$$

Therefore, AZ bisects the angle between lines  $m$  and  $n$ .

**Example 2:** In a  $\Delta ABC$ ,  $BD = DC$ . If the perpendiculars from point D to sides AB and AC are equal, then prove that  $AB = AC$ .

**Solution:**

The triangle with the given specifications is drawn below.



In  $\Delta ABC$ , D is the midpoint of BC. Also, DE and DF are the perpendiculars from D to AB and AC respectively.

In  $\Delta DEB$  and  $\Delta DFC$ , we have:

$$\angle DEB = \angle DFC = 90^\circ \ (\because DE \text{ and } DF \text{ are perpendiculars})$$

$$DB = DC \text{ (Given)}$$

$$DE = DF \text{ (Given)}$$

So, by the RHS congruence rule, we obtain:

$$\triangle DEB \cong \triangle DFC$$

$$\Rightarrow \angle DBE = \angle DCF \text{ (By CPCT)}$$

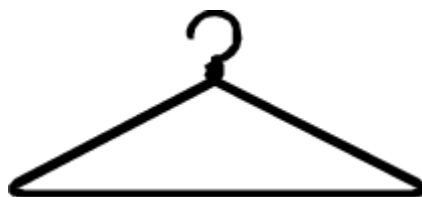
$$\Rightarrow \angle CBA = \angle BCA$$

$$\Rightarrow AC = AB \text{ (As sides opposite to equal angles are equal)}$$

### Angles Opposite to Equal Sides of an Isosceles Triangle are Equal

#### Isosceles Triangles around Us

You must have seen triangular clothes hangers or coat hangers such as the one shown.



What do you observe about the sides of the hanger? The base is clearly the longest side, while the other two sides are equal. We know that a triangle with any two sides equal is called an isosceles triangle. So, the hanger is a real-life example of an isosceles triangle. Another example of an isosceles triangle is the roof of a hut. Yet another example is a diagonally cut slice of bread.

You can see that the angles opposite the equal sides of the clothes hanger are equal. In this lesson, we will discuss about the same, i.e., angles opposite equal sides of an isosceles triangle. We will also solve some problems based on this concept.

#### Know Your Scientist



Thales (624 BC–546 BC) is believed to be the first philosopher in the Greek tradition. A great mathematician and scientist, he is regarded as 'the father of science'. Thales is credited for giving the first known mathematical proof, i.e., 'a circle is bisected by its diameter'. He was also the first to prove the theorem: 'any angle inscribed in a semicircle is a right angle'. For this reason, this theorem is called Thales Theorem.

Thales also generalized the concept of equality of the base angles of an isosceles triangle. He also propounded a few general notions such as 'all straight angles are equal', 'equals subtracted from equals are equal' and 'equals added to equals are equal'. These theorems have helped in the evolution of modern mathematics.



### Did You Know?

The word 'isosceles' is a combination of the Greek words 'isos' meaning 'equal' and 'skelos' meaning 'leg'. So, a triangle having two equal legs is called an isosceles triangle.

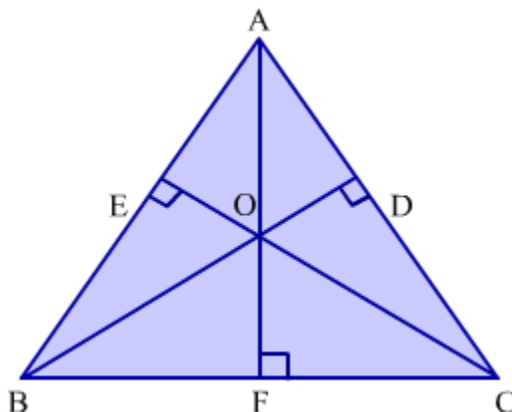
### Know More

- The altitude to the base of an isosceles triangle bisects the vertex angle.
- The altitude to the base of an isosceles triangle bisects the base.
- When the equal sides are at right angle, the triangle is called a 'right isosceles triangle'.

### Whiz Kid

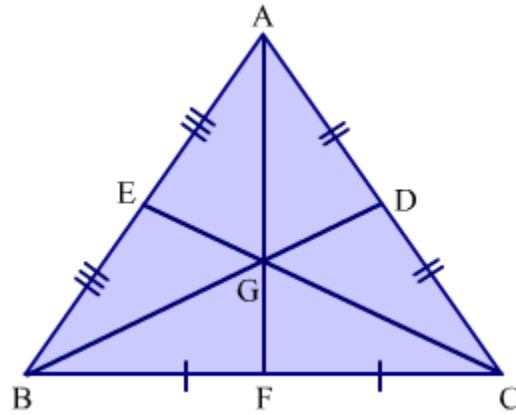
In an isosceles triangle, the orthocentre, the centroid, the incentre and the circumcentre, all lie on the median to the base.

**Orthocentre:** It is the point where the altitudes of the three sides of a triangle intersect.



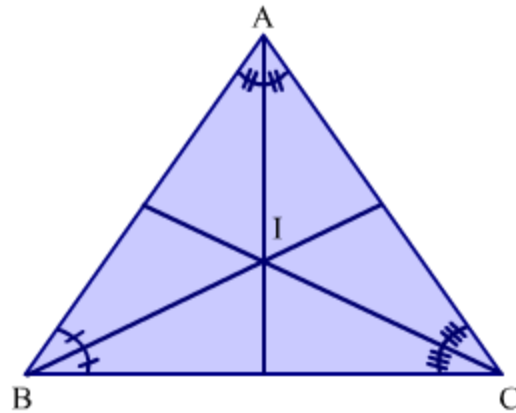
In  $\triangle ABC$ , O is the orthocentre.

**Centroid:** It is the point where the medians of the three sides of a triangle intersect.



In  $\triangle ABC$ , G is the centroid.

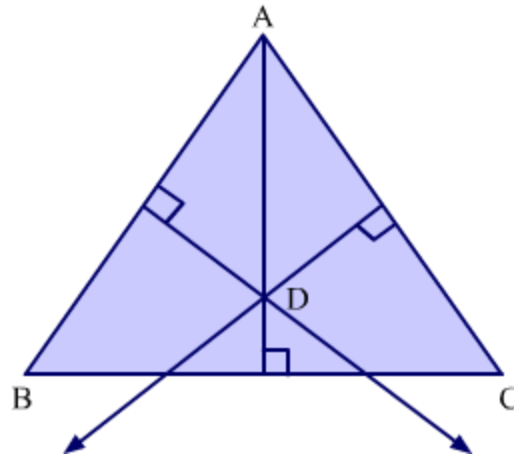
**Incentre:** It is the point where the interior angle bisectors of a triangle intersect.



In  $\triangle ABC$ , I is the incentre.

**Circumcentre:** It is the point where the perpendicular bisectors of the three sides of a triangle intersect.



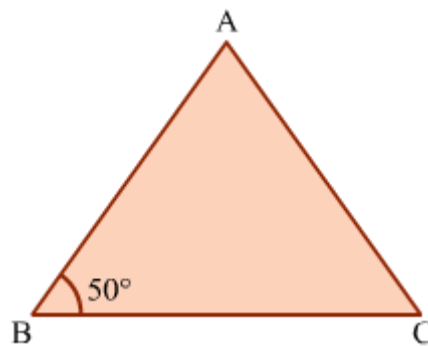


In  $\triangle ABC$ , D is the circumcentre.

### Solved Examples

Easy

**Example 1:** In the given  $\triangle ABC$ ,  $AB = AC$ . What is the measure of  $\angle BAC$ ?



**Solution:**

$\triangle ABC$  is an isosceles triangle with  $AB = AC$ .

By the property of isosceles triangles, we obtain:

$$\angle ABC = \angle ACB$$

$$\Rightarrow \angle ACB = 50^\circ$$

By the angle sum property, we have:

$$\angle ABC + \angle ACB + \angle BAC = 180^\circ$$

$$\Rightarrow 50^\circ + 50^\circ + \angle BAC = 180^\circ$$

$$\Rightarrow \angle BAC = 180^\circ - 100^\circ$$

$$\Rightarrow \angle BAC = 80^\circ$$

**Example 2:** Ram lives in a triangular tree house. The measure of each base angle is the same and the length of the base is 4 m. He wants to cover the slant sides with metal sheets. If the boundary of the triangular structure is 10 m, then what length of sheet does Ram need to cover the slant sides (irrespective of the width of the sheet).

**Solution:**

It is given that the base angles are equal; so, the triangular structure is isosceles. Hence, the slant sides are equal.

Now, we know that:

$$\text{Base} = 4 \text{ m}$$

$$\text{Boundary of the triangular structure} = \text{Perimeter of the isosceles triangle} = 10 \text{ m}$$

$$\Rightarrow 10 \text{ m} = 4 \text{ m} + 2 \times (\text{Measure of each slant side})$$

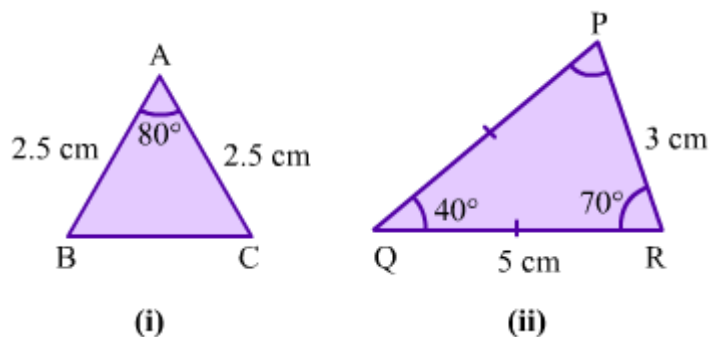
$$\Rightarrow 2 \times (\text{Measure of each slant side}) = 10 \text{ m} - 4 \text{ m}$$

$$\Rightarrow 2 \times (\text{Measure of each slant side}) = 6 \text{ m}$$

Hence, Ram needs 6 m of metal sheet to cover the slant sides of the triangular structure.

**Medium**

**Example 1:** Find the missing angles in the following triangles.



**Solution:**

1. In  $\triangle ABC$ ,  $AB = AC = 2.5$  cm

Since the angles opposite equal sides of a triangle are equal, we obtain:

$$\angle ABC = \angle ACB$$

$$\text{Let } \angle ABC = \angle ACB = x$$

By the angle sum property of triangles, we have:

$$\angle ABC + \angle ACB + \angle BAC = 180^\circ$$

$$\Rightarrow x + x + 80^\circ = 180^\circ$$

$$\Rightarrow 2x = 100^\circ$$

$$\Rightarrow x = 50^\circ$$

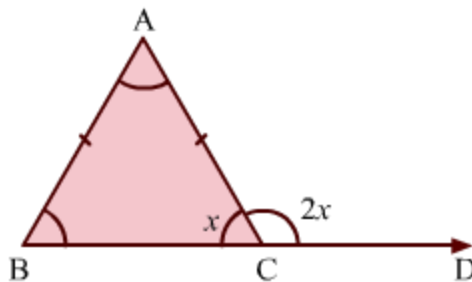
$$\text{Thus, } \angle ABC = \angle ACB = 50^\circ$$

2. In  $\triangle PQR$ ,  $PQ = QR = 5$  cm and  $PR = 3$  cm

Since  $PQ = QR$ , we obtain

$$\angle QPR = \angle PRQ = 70^\circ$$

**Example 2:** The shown  $\triangle ABC$  is isosceles with  $AB = AC$ . Find the measures of  $\angle BAC$ ,  $\angle ABC$  and  $\angle ACB$ .



**Solution:**

$\angle ACB$  and exterior angle  $\angle ACD$  form a linear pair.

$$\therefore \angle ACB + \angle ACD = 180^\circ$$

$$\Rightarrow x + 2x = 180^\circ$$

$$\Rightarrow 3x = 180^\circ$$

$$\Rightarrow \therefore x = 60^\circ$$

So,  $\angle ACB = 60^\circ$  and  $\angle ACD = 120^\circ$

$\triangle ABC$  is isosceles.

$\therefore \angle ABC = \angle ACB = 60^\circ$  ( $\because$  Angles opposite equal sides are equal)

Now, by the exterior angle property, we have:

$$\angle ACD = \angle BAC + \angle ABC$$

$$\Rightarrow 120^\circ = \angle BAC + 60^\circ$$

$$\Rightarrow \angle BAC = 60^\circ$$

Thus,  $\angle BAC = \angle ABC = \angle ACB = 60^\circ$ .

**Example 3:** Prove that in an isosceles triangle, the angle bisector of the apex angle is the perpendicular bisector of the base.

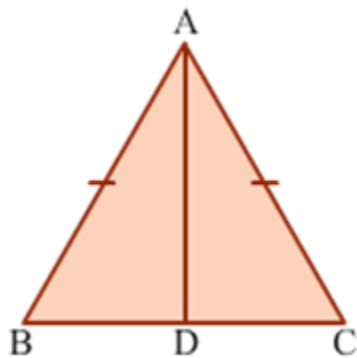
**Solution:**

**Given:** ABC is a triangle in which  $AB = AC$  and apex angle  $\angle A$ .

**To Prove:** AD is perpendicular bisector of BC and  $BD = DC$ .

**Construction:** Draw an angle bisector AD from A on BC.

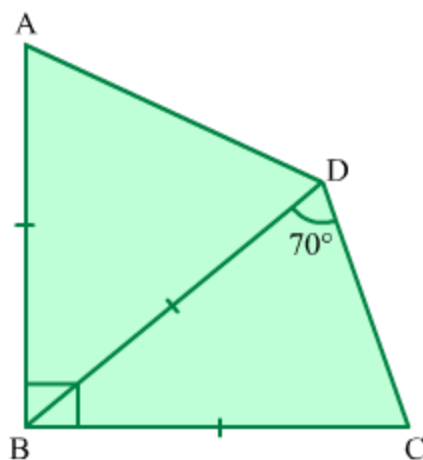
**Proof:**



In  $\triangle ABD$  and  $\triangle ACD$ ,  
 $AB = AC$  (Given)  
 $AD = AD$  (Common side)  
 $\angle BAD = \angle CAD$  (AD is a bisector of  $\angle A$ )  
 So, by SAS congruence criterion,  
 $\triangle ABD \cong \triangle ACD$   
 $\Rightarrow BD = DC$  and  $\angle ADB = \angle ADC$  (CPCT)  
 Now,  
 $\angle ADB + \angle ADC = 180^\circ$  (Linear Pair)  
 $\Rightarrow \angle ADB + \angle ADB = 180^\circ$   
 $\Rightarrow 2\angle ADB = 180^\circ$   
 $\Rightarrow \angle ADB = \frac{180^\circ}{2} = 90^\circ$   
 Thus, AD is the perpendicular bisector of BC.

### Hard

**Example 1:**  $\triangle ABD$  is isosceles (figure not drawn to scale) with  $AB = BD$ , while  $\triangle BCD$  is isosceles with  $BC = BD$ . Also,  $\angle BDC$  measures  $70^\circ$  and  $\angle ABC$  measures  $90^\circ$ . What is the measure of  $\angle ADC$ ?



**Solution:**

In  $\triangle BCD$ ,  $BC = BD$ .

$\therefore \angle BCD = \angle BDC = 70^\circ$  ( $\because$  Angles opposite equal sides are equal)

In  $\triangle BCD$ , by the angle sum property, we obtain:

$$\angle BCD + \angle BDC + \angle CBD = 180^\circ$$

$$\Rightarrow 70^\circ + 70^\circ + \angle CBD = 180^\circ$$

$$\Rightarrow 140^\circ + \angle CBD = 180^\circ$$

$$\Rightarrow \angle CBD = 40^\circ$$

It is given that  $\angle ABC = 90^\circ$ .

$$\Rightarrow \angle ABD + \angle CBD = 90^\circ$$

$$\Rightarrow \angle ABD + 40^\circ = 90^\circ$$

$$\Rightarrow \angle ABD = 50^\circ$$

In  $\triangle ABD$ , we have:

$$AB = BD$$

$\therefore \angle BAD = \angle BDA = x$  ( $\because$  Angles opposite equal sides are equal)

In  $\triangle ABD$ , by the angle sum property, we obtain:

$$\angle ABD + \angle BAD + \angle BDA = 180^\circ$$

$$\Rightarrow 50^\circ + x + x = 180^\circ$$

$$\Rightarrow 2x = 130^\circ$$

$$\Rightarrow x = 65^\circ$$

$$\text{So, } \angle BAD = \angle BDA = 65^\circ$$

Now,

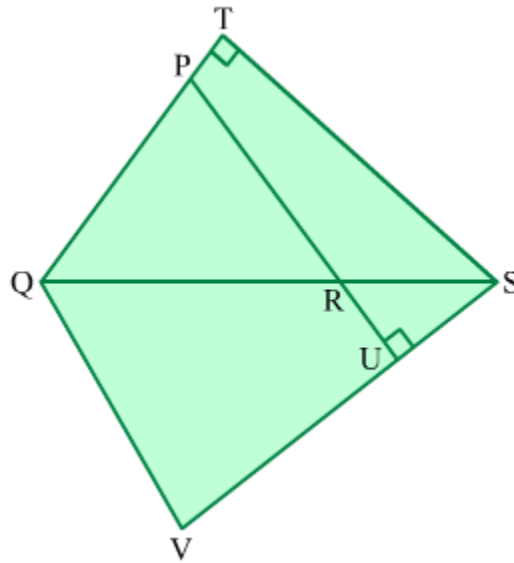


$$\angle ADC = \angle BDA + \angle CDB$$

$$\Rightarrow \angle ADC = 65^\circ + 70^\circ$$

$$\Rightarrow \angle ADC = 135^\circ$$

**Example 2:** In the given figure,  $\Delta PQR$  is isosceles with  $PQ = PR$ . Side  $SU$  is extended up to point  $V$  such that  $ST = SV$ . If  $\angle QTS = \angle RUS = 90^\circ$ , prove that  $QS$  bisects  $\angle TSU$  and hence show that  $\Delta QTS \cong \Delta QVS$ .



**Solution:**

In  $\Delta PQR$ , we have:

$$PQ = PR \text{ (Given)}$$

$$\therefore \angle PQR = \angle PRQ \text{ (}\because \text{Angles opposite equal sides are equal)}$$

It is given that  $\angle QTS = 90^\circ$ .

Using the angle sum property in  $\Delta QTS$ , we obtain:

$$\angle TQS + \angle TSQ + \angle QTS = 180^\circ$$

$$\Rightarrow \angle TQS + \angle TSQ + 90^\circ = 180^\circ$$

$$\Rightarrow \angle TSQ = 90^\circ - \angle TQS \dots (1)$$

It is given that  $\angle RUS = 90^\circ$ .

Using the angle sum property in  $\triangle RUS$ , we obtain:

$$\angle SRU + \angle RUS + \angle RSU = 180^\circ$$

$$\Rightarrow \angle SRU + 90^\circ + \angle RSU = 180^\circ$$

$$\Rightarrow \angle RSU = 90^\circ - \angle SRU \dots (2)$$

$$\angle SRU = \angle PRQ \text{ (Vertically opposite angles)}$$

$$\text{Also, } \angle PQR = \angle PRQ$$

$$\therefore \angle SRU = \angle PQR \dots (3)$$

Now, from equations (2) and (3), we get:

$$\angle RSU = 90^\circ - \angle PQR$$

$$\Rightarrow \angle RSU = 90^\circ - \angle TQS \dots (4) \quad [\because \angle PQR = \angle TQS]$$

From equations (1) and (4), we get:

$$\angle TSQ = \angle RSU$$

Hence, QS bisects  $\angle TSU$ .

Now, consider  $\triangle QTS$  and  $\triangle QVS$ .

$$QS = QS \text{ (Common side)}$$

$$\angle TSQ = \angle QSV \quad (\because \angle TSQ = \angle RSU \text{ and } \angle RSU = \angle QSV)$$

$$ST = SV \text{ (Given)}$$

Thus, by the SAS congruency rule, we get:

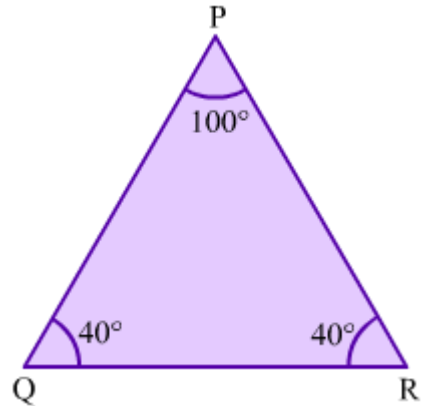
$$\triangle QTS \cong \triangle QVS$$

**Sides Opposite to Equal Angles of a Triangle are Equal**

**Observing the Equal Angles and the Sides Opposite to Them in an Isosceles Triangle**

Consider the following  $\triangle PQR$ .





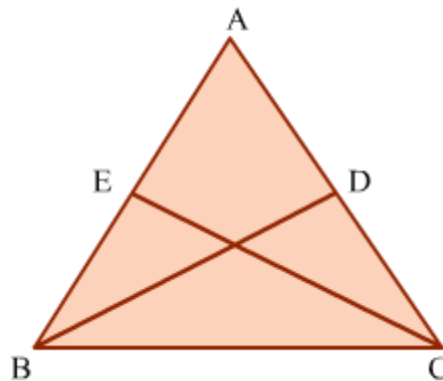
Is  $\Delta PQR$  isosceles? We know that if two sides of a triangle are equal (or congruent), then the triangle is isosceles. However, in  $\Delta PQR$ , two angles are equal (or congruent). We have studied that the angles opposite to equal (or congruent) sides of an isosceles triangle are equal (or congruent). Is the converse of this property also true?

In this lesson, we will study about the equality of sides opposite equal angles in an isosceles triangle. We will also solve some examples related to this concept.

### Whiz Kid

In an isosceles triangle, the medians drawn from the base vertices to the opposite sides are of equal length.

For example:



The shown  $\Delta ABC$  is isosceles such that  $AB = AC$ .  $BD$  and  $CE$  are the respective medians from vertices B and C to sides AC and AB. Therefore,  $BD = CE$ .

### Solved Examples

Easy

**Example 1:** In a  $\triangle ABC$ ,  $\angle BAC = 2x$  and  $\angle ABC = \angle ACB = x$ . Find the value of  $x$  and hence show that  $AB = AC$ .

**Solution:**

It is given that  $\angle BAC = 2x$  and  $\angle ABC = \angle ACB = x$ .

By applying the angle sum property, we obtain:

$$\angle BAC + \angle ABC + \angle ACB = 180^\circ$$

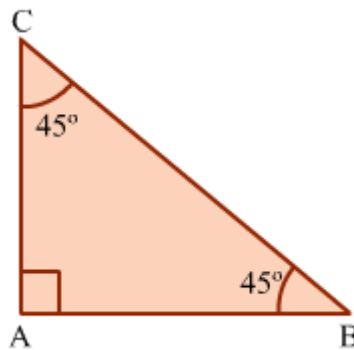
$$\Rightarrow 2x + x + x = 180^\circ$$

$$\Rightarrow 4x = 180^\circ$$

$$\Rightarrow x = 45^\circ$$

So,  $\angle BAC = 2x = 2 \times 45^\circ = 90^\circ$  and  $\angle ABC = \angle ACB = x = 45^\circ$

The given triangle can be drawn as is shown.

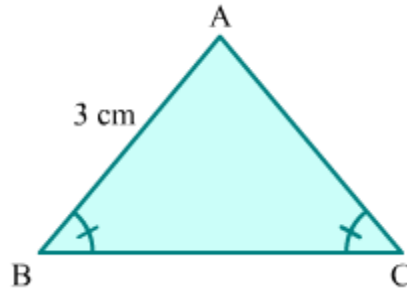


We know that the sides opposite equal angles of a triangle are equal.

$$\therefore AB = AC$$

Hence,  $\triangle ABC$  is isosceles with  $AB = AC$ .

**Example 2:** In the given  $\triangle ABC$ ,  $\angle ABC = \angle ACB$  and the perimeter is 11 cm. Find the length of the base of the triangle.



**Solution:** It is given that  $\angle ABC = \angle ACB$  and  $AB = 3$  cm.

We know that the sides opposite equal angles of a triangle are equal.

$$\therefore AC = AB = 3 \text{ cm}$$

$$\text{Perimeter of } \triangle ABC = AB + BC + AC$$

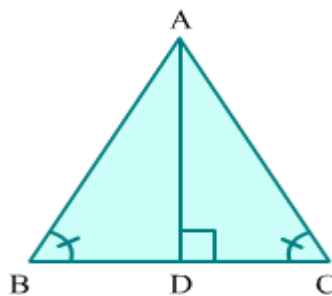
$$\Rightarrow 11 \text{ cm} = 3 \text{ cm} + BC + 3 \text{ cm}$$

$$\Rightarrow BC = 5 \text{ cm}$$

Thus, the length of the base of  $\triangle ABC$  is 5 cm.

### Medium

**Example 1:** In the given  $\triangle ABC$ ,  $\angle ABD = \angle ACD$  and  $AD$  is perpendicular to  $BC$ . Prove that  $AD$  bisects  $\angle BAC$ .



**Solution:**

In  $\triangle ABD$  and  $\triangle ACD$ , we have:

$$\angle ABD = \angle ACD \text{ (Given)}$$

$$\angle ADB = \angle ADC = 90^\circ$$

$AB = AC$  ( $\because$  Sides opposite equal angles are equal)

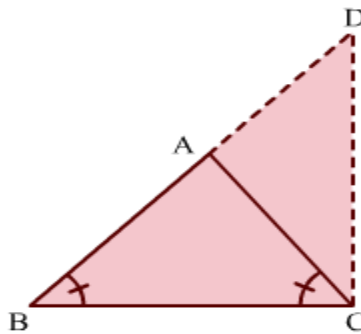
Thus, by the AAS congruence rule, we have:

$$\triangle ABD \cong \triangle ACD$$

$$\therefore \angle BAD = \angle CAD \text{ (By CPCT)}$$

Thus, AD bisects  $\angle BAC$ .

**Example 2:** In the given  $\triangle ABC$ ,  $\angle ABC = \angle ACB$ . Side BA is produced up to point D such that  $AB = AD$ . Prove that  $\angle BCD$  is a right angle.



**Solution:**

In  $\triangle ABC$ , we have:

$$\angle ABC = \angle ACB \dots (1) \text{ [Given]}$$

$$\therefore AB = AC \text{ } (\because \text{Sides opposite equal angles are equal})$$

Now,

$$AB = AD \text{ (Given)}$$

$$AD = AC \text{ } (\because AB = AC)$$

Thus, in  $\triangle ADC$ , we have:

$$\angle ACD = \angle ADC \dots (2) \text{ } (\because \text{Angles opposite equal sides are equal})$$

On adding equations (1) and (2), we get:

$$\angle ACB + \angle ACD = \angle ABC + \angle ADC$$

$$\Rightarrow \angle BCD = \angle ABC + \angle BDC \quad (\because \angle ADC = \angle BDC)$$

On adding  $\angle BCD$  to both sides of the equation, we get:

$$\angle BCD + \angle BCD = \angle ABC + \angle BDC + \angle BCD$$

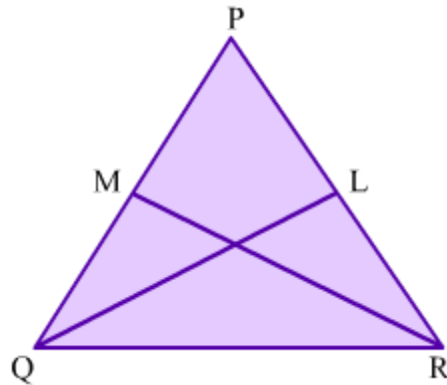
$$\Rightarrow 2\angle BCD = 180^\circ \quad (\text{By the angle sum property})$$

$$\Rightarrow \angle BCD = 90^\circ$$

Hence,  $\angle BCD$  is a right angle.

### Hard

**Example 1:** The shown  $\triangle PQR$  is isosceles with  $PQ = PR$ .  $QL$  and  $RM$  are the respective medians from vertices  $Q$  and  $R$  to sides  $PR$  and  $PQ$ . Prove that the medians have the same length.



### Solution:

It is given that  $PQ = PR$ .

Also,  $QL$  and  $RM$  are medians.

$$\therefore PL = LR \text{ and } PM = MQ$$

So,

$$PL + LR = PR$$

$$LR + LR = PR$$

$$2LR = PR$$

$$LR = \frac{PR}{2} \dots (1)$$

$$\text{Similarly, } MQ = \frac{PQ}{2} \dots (2)$$

Since  $PQ = PR$ , using equations (1) and (2), we obtain:

$$LR = MQ \dots (3)$$

In  $\triangle QRL$  and  $\triangle RQM$ , we have:

$$LR = MQ \text{ (From (3))}$$

$$\angle LRQ = \angle MQR \text{ } (\because \text{Angles opposite equal sides are equal})$$

$$QR = RQ \text{ (Common side)}$$

$$\therefore \triangle QRL \cong \triangle RQM \text{ (By the SAS congruence criterion)}$$

$$\Rightarrow QL = RM \text{ (By CPCT)}$$

Thus, the medians  $QL$  and  $RM$  have the same length.

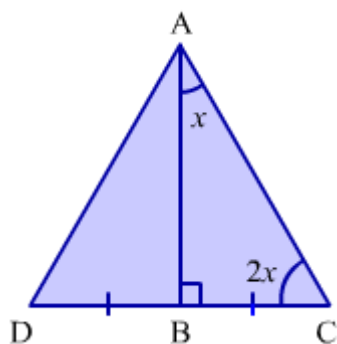
### Example 2:

In a right-angled triangle  $ABC$ ,  $\angle ACB = 2\angle BAC$ . Prove that  $AC = 2BC$ .

### Solution:

The given  $\triangle ABC$  can be drawn as is shown.

**Construction:** Produce  $CB$  up to point  $D$  such that  $BD = BC$ . Join point  $A$  to point  $D$ .



In  $\triangle ABD$  and  $\triangle ABC$ , we have:

$$BD = BC \text{ (By construction)}$$

$$\angle ABD = \angle ABC = 90^\circ$$

$$AB = AB \text{ (Common side)}$$

So, by the SAS congruence rule, we obtain:

$$\triangle ABD \cong \triangle ABC$$

$$\Rightarrow AD = AC \text{ and } \angle DAB = \angle BAC \text{ (By CPCT)}$$

Let  $\angle BAC$  be  $x$ . Then,  $\angle DAB$  will also be  $x$ .

$$\text{Now, } \angle DAC = \angle DAB + \angle BAC$$

$$\Rightarrow \angle DAC = x + x$$

$$\Rightarrow \angle DAC = 2x$$

$$\Rightarrow \angle DAC = \angle ACB \text{ } (\because \angle ACB = 2\angle BAC = 2x)$$

$$\Rightarrow DC = AD \text{ } (\because \text{Sides opposite equal angles are equal})$$

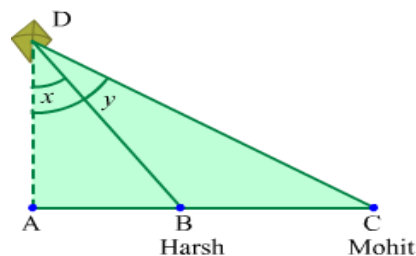
Since  $BC = DB$ , we have:

$$DC = 2BC$$

$$\Rightarrow 2BC = AD$$

$$\Rightarrow 2BC = AC \text{ } (\because \text{We have proved } AD = AC)$$

**Observing the Relation between the Angles and Sides of a Triangle**



Harsh is flying a kite. The thread of the kite makes angle  $x$  with the vertical, as is shown in the figure. Sometime later, he gives the thread to his friend Mohit who is standing some distance away from him. At that position, the thread makes angle  $y$  with the vertical.

The figure clearly shows that angle  $y$  of  $\triangle ACD$  is greater than angle  $x$  of  $\triangle ABD$ . Also, side  $AC$  is greater than side  $AB$ . This shows us that if we increase the length of any side of a triangle, then the angle facing that side also increases.

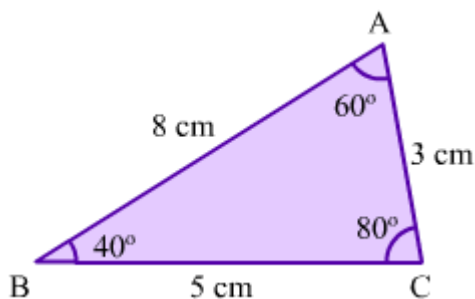
In this lesson, we will discuss the relation between the angles and sides of a triangle. We will then solve some examples relating to the same.

### Triangle Inequality Theorem

We know that if two sides of a triangle are equal, then the angles opposite these sides are also equal. Now, what **if all the sides of a triangle are unequal? What can be said about its angles?** The **triangle inequality theorem** describes such a triangle. It states that:

***If two sides of a triangle are unequal, then the longer side has the greater angle opposite it.***

Consider the given  $\triangle ABC$ .



Let us apply the stated theorem in this triangle.

$AB = 8$  cm is the longest side in  $\triangle ABC$ . Therefore, the angle opposite  $AB$ , i.e.,  $\angle BCA$  is the greatest angle of the triangle. Also,  $AC = 3$  cm is the shortest side in  $\triangle ABC$ . Therefore, the angle opposite  $AC$ , i.e.,  $\angle ABC$  is the smallest angle of the triangle.

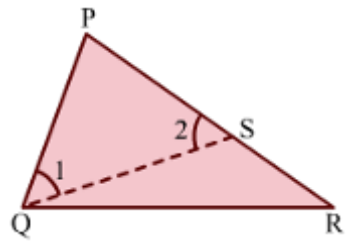
Thus, we can conclude that the angle opposite the longer side is greater. Therefore, the theorem holds true.

### Proving the Triangle Inequality Theorem

**Given:**  $\triangle PQR$  in which  $PR > PQ$



**To prove:**  $\angle Q > \angle R$



**Construction:** Mark a point S on PR such that  $PQ = PS$ . Join Q to S.

**Proof:** In  $\triangle PQS$ , we have:

$PQ = PS$  (By construction)

$\Rightarrow \angle 1 = \angle 2 \dots (1)$  [ $\because$  Angles opposite equal sides are equal]

In  $\triangle QRS$ ,  $\angle 2$  is the exterior angle; so, it is greater than the interior opposite angles of  $\triangle QRS$ .

$\therefore \angle 2 > \angle SRQ$

$\Rightarrow \angle 2 > \angle PRQ \dots (2)$  [ $\because \angle PRQ = \angle SRQ$ ]

From (1) and (2), we have:

$\angle 1 > \angle PRQ \dots (3)$

Now,  $\angle 1$  is a part of  $\angle PQR$ .

So,  $\angle PQR > \angle 1 \dots (4)$

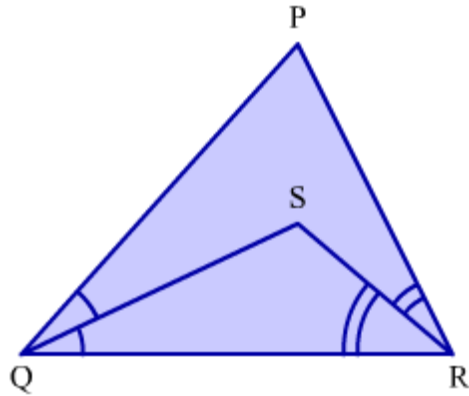
Thus, from (3) and (4), we can conclude that:

$\angle PQR > \angle PRQ$

### Solved Examples

#### Easy

**Example 1:** In the given  $\triangle PQR$ ,  $PQ$  is greater than  $PR$ . Also,  $QS$  and  $RS$  are the respective bisectors of  $\angle PQR$  and  $\angle PRQ$ . Prove that  $\angle SRQ > \angle SQR$ .



**Solution:**

In  $\Delta PQR$ , we have:

$PQ > PR$  (Given)

$\Rightarrow \angle PRQ > \angle PQR$  ( $\because$  Angle opposite longer side is greater)

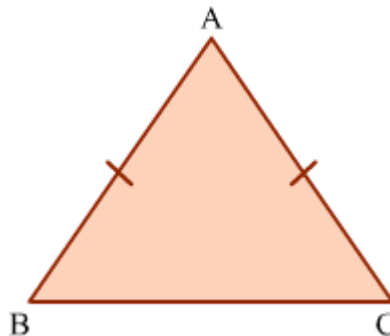
$$\Rightarrow \frac{\angle PRQ}{2} > \frac{\angle PQR}{2}$$

$\Rightarrow \angle SRQ > \angle SQR$  ( $\because$  RS bisects  $\angle PRQ$  and QS bisects  $\angle PQR$ )

**Example 2:** ABC is an isosceles triangle with  $AB = AC$  and  $AB < BC$ . Prove that  $\angle BAC > \angle ABC$ .

**Solution:**

Consider the following  $\Delta ABC$  in which  $AB = AC$  and  $AB < BC$ .



In  $\Delta ABC$ , we have:

$\angle ABC = \angle ACB \dots (1)$  [ $\because$  Angles opposite equal sides AB and AC are equal]

By the triangle inequality theorem, we have:

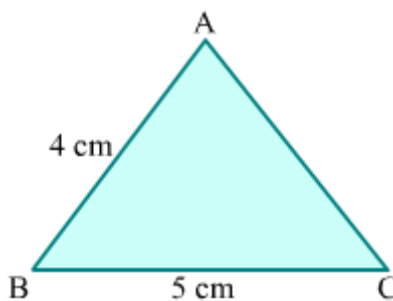
$\angle ACB < \angle BAC \dots (2)$  [ $\because$  Angle opposite longer side BC is greater]

Thus, from (1) and (2), we obtain:

$\angle BAC > \angle ABC$

### Medium

**Example 1:** In the shown  $\triangle ABC$ ,  $AB = 4$  cm,  $BC = 5$  cm and the perimeter is 16 cm. Determine the smallest and greatest angles of the triangle.



### Solution:

In  $\triangle ABC$ , we have:

$AB = 4$  cm and  $BC = 5$  cm (Given)

Perimeter = 16 cm (Also given)

$\Rightarrow AB + BC + CA = 16$  cm ( $\because$  Perimeter is the sum of all sides)

$\Rightarrow 4$  cm +  $5$  cm +  $CA = 16$  cm

$\Rightarrow CA = 7$  cm

Now,  $CA = 7$  cm is the longest side of  $\triangle ABC$ . Thus, the angle opposite it, i.e.,  $\angle ABC$  is the greatest angle of the triangle.

$\therefore \angle ABC > \angle BAC$  and  $\angle ABC > \angle ACB \dots (1)$

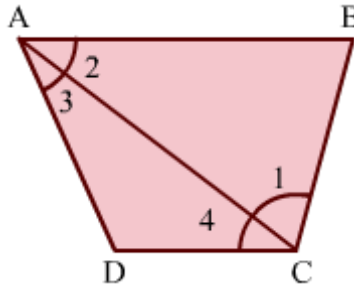
Also,  $BC > AB$



$$\therefore \angle BAC > \angle ACB \dots (2)$$

Thus, from (1) and (2), we can conclude that  $\angle ABC$  is the greatest angle and  $\angle ACB$  is the smallest angle in  $\triangle ABC$ .

**Example 2:** Suppose AB is the longest side and CD the shortest side of the given quadrilateral ABCD. Then, prove that  $\angle BCD > \angle BAD$ .



**Solution:**

In  $\triangle ABC$ , AB is the longest side.

So,  $AB > BC$

$$\Rightarrow \angle 1 > \angle 2 \dots (1) \quad [\because \text{Angle opposite longer side is greater}]$$

In  $\triangle ADC$ , CD is the shortest side.

So,  $AD > CD$

$$\Rightarrow \angle 4 > \angle 3 \dots (2) \quad [\because \text{Angle opposite longer side is greater}]$$

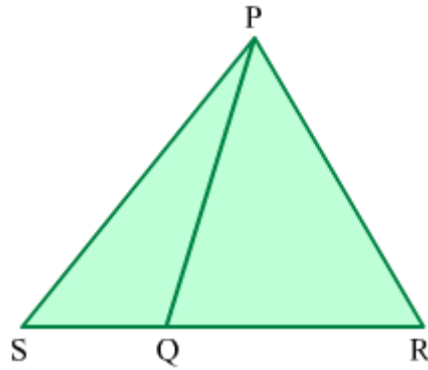
On adding (1) and (2), we get:

$$\angle 1 + \angle 4 > \angle 2 + \angle 3$$

$$\Rightarrow \angle BCD > \angle BAD$$

**Hard**

**Example 1:** In the given figure,  $PQ = PR$ . Show that  $\angle PRS > \angle PSR$ .



**Solution:**

In  $\triangle PQR$ , we have:

$$PQ = PR$$

$$\Rightarrow \angle PRQ = \angle PQR \dots (1) \quad [\because \text{Angles opposite equal sides are equal}]$$

In  $\triangle PSQ$ , SQ is produced to R.

$$\text{So, } \angle PQR = \angle PSQ + \angle SPQ \quad (\text{By the exterior angle property})$$

$$\Rightarrow \angle PQR > \angle PSQ \dots (2)$$

From (1) and (2), we obtain:

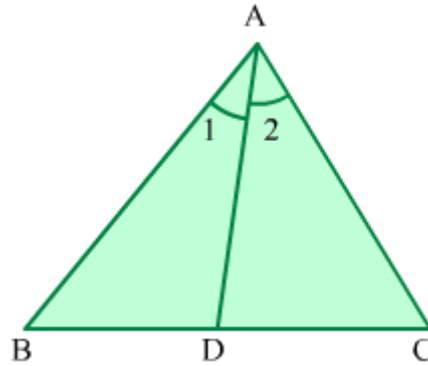
$$\angle PRQ > \angle PSQ$$

$$\Rightarrow \angle PRS > \angle PSR \quad (\because \angle PRQ = \angle PRS \text{ and } \angle PSQ = \angle PSR)$$

**Example 2:** In a  $\triangle ABC$ ,  $AB > AC$  and AD is the bisector of  $\angle BAC$ . Show that  $\angle ADB > \angle ADC$ .

**Solution:**

Let the following  $\triangle ABC$  be the given triangle such that  $AB > AC$ . Also, AD is the bisector of  $\angle BAC$ .



In  $\triangle ABC$ , we have:

$AB > AC$  (Given)

$\Rightarrow \angle ACB > \angle ABC$  ( $\because$  Angle opposite longer side is greater)

On adding  $\angle 1$  to both sides, we get:

$$\angle ACB + \angle 1 > \angle ABC + \angle 1$$

$$\Rightarrow \angle ACB + \angle 2 > \angle ABC + \angle 1 \dots (1) \quad [\because AD \text{ bisects } \angle BAC; \angle 1 = \angle 2]$$

By the exterior angle property, we have:

$$\angle ADB = \angle ACB + \angle 2 \dots (2)$$

$$\text{Similarly, } \angle ADC = \angle ABC + \angle 1 \dots (3)$$

Thus, by using (1), (2) and (3), we can conclude that:

$$\angle ADB > \angle ADC$$

### Observation of the Sides of a Triangle by Seeing the Angles

The red car has to travel at an angle of  $65^\circ$ , while the blue car has to travel at an angle of  $50^\circ$  with respect to line AB to reach the finish point C. Do you think any one car has an advantage over the other?

On observing the triangular track, it seems that path BC is longer than path AC. So, clearly, the red car has an advantage over the blue car. Also, the angle opposite BC is greater than the angle opposite AC. So, what does this tell us about the relation between the sides and angles of the triangular racetrack?

Let us go through this lesson to learn about the relation between the sides and angles of a triangle. We will also solve some problems based on this relation.

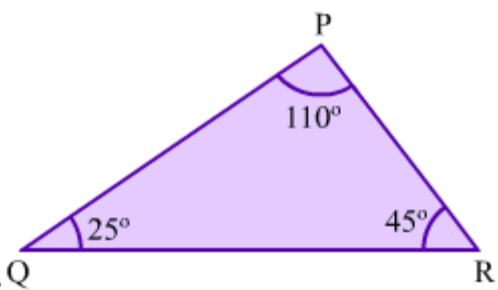
### **In a Triangle, the Side Opposite the Greater Angle is**

#### **Longer**

We have studied that in a triangle having two unequal sides, the angle opposite the longer side is greater. The converse of this property is also true. It states that:

***If two angles of a triangle are unequal, then the greater angle has the longer side opposite it. In other words, the smaller angle has the shorter side opposite it.***

Consider the following  $\triangle PQR$ .



Let us apply the stated property in this triangle  $\triangle PQR$ .

$\angle QPR = 110^\circ$  is the greatest angle in  $\triangle PQR$ .

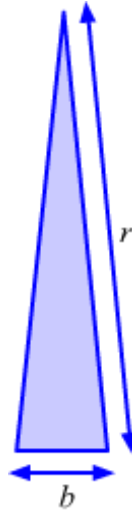
Therefore, the side opposite  $\angle QPR$ , i.e.,  $QR$  is the longest side of the triangle. Also,  $\angle PQR = 25^\circ$  is the smallest angle in  $\triangle PQR$ .

Therefore, the side opposite  $\angle PQR$ , i.e.,  $PR$  is the shortest side of the triangle.

We can apply this property to other triangles as well. Using this property, we can say that in a right triangle, the hypotenuse is the longest side.

#### **Whiz Kid**

In trigonometry, a skinny triangle is one whose height is much greater than its base.



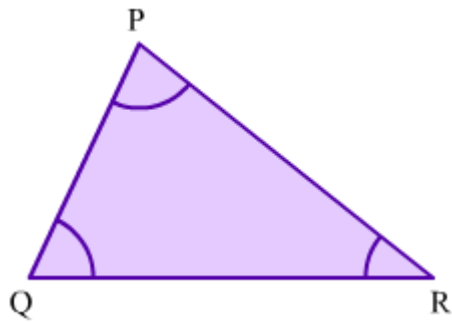
The given triangle is an example of a skinny triangle.

### Proving the Property

**Given:**  $\Delta PQR$  in which  $\angle Q > \angle R$

**To prove:**  $PR > PQ$ .

**Proof:** In  $\Delta PQR$ , we can have three possible cases: (1)  $PQ > PR$ , (2)  $PQ = PR$  and (3)  $PQ < PR$ .



#### CASE 1:

When  $PQ > PR$ , we have:

$\angle R > \angle Q$  ( $\because$  Angle opposite longer side is greater)

But this contradicts the given hypothesis that  $\angle R < \angle Q$ . Thus,  $PQ > PR$  is not true.

#### CASE 2:

When  $PQ = PR$ , we have:



$\angle R = \angle Q$  ( $\because$  Angles opposite equal sides are equal)

But this too contradicts the given hypothesis that  $\angle R < \angle Q$ . Thus,  $PQ = PR$  is also not true.

So, we are left with the third possibility, i.e.,  $PQ < PR$  (or  $PR > PQ$ ), which must be true.

Thus, we have proven that if two angles of a triangle are unequal, then the greater angle has the longer side opposite it.

## Solved Examples

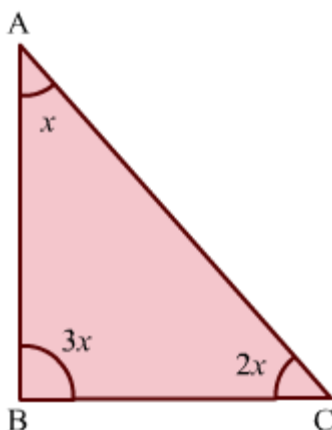
### Easy

**Example 1:** The angles of a triangle are in the ratio 1 : 2 : 3. Find the greatest angle and identify the longest side of the triangle.

**Solution:**

It is given that the angles of the triangle, say  $\triangle ABC$ , are in ratio 1 : 2 : 3.

Let the angles be  $x$ ,  $2x$  and  $3x$ , as is shown in the figure.



By the angle sum property of triangles, we have:

$$\angle BAC + \angle ABC + \angle ACB = 180^\circ$$

$$\Rightarrow x + 3x + 2x = 180^\circ$$

$$\Rightarrow 6x = 180^\circ$$

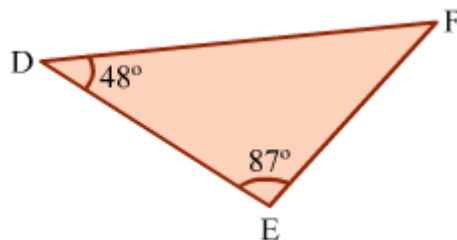
$$\Rightarrow x = 30^\circ$$

So,  $2x = 2 \times 30^\circ = 60^\circ$  and  $3x = 3 \times 30^\circ = 90^\circ$

Thus, the greatest angle in the given triangle is  $\angle ABC$ , i.e.,  $90^\circ$ .

Now, we know that the side opposite the greater angle is longer. In  $\triangle ABC$ , side AC is opposite the greatest angle; hence, it is the longest.

**Example 2:** Which side of the given triangle is the shortest?



**Solution:**

In order to find the shortest side of  $\triangle DEF$ , we need to figure out the smallest angle of the triangle. This is because the smallest side is opposite the smallest angle.

By the angle sum property of triangles, we have:

$$\angle EDF + \angle DEF + \angle EFD = 180^\circ$$

$$\Rightarrow 48^\circ + 87^\circ + \angle EFD = 180^\circ$$

$$\Rightarrow 135^\circ + \angle EFD = 180^\circ$$

$$\Rightarrow \angle EFD = 45^\circ$$

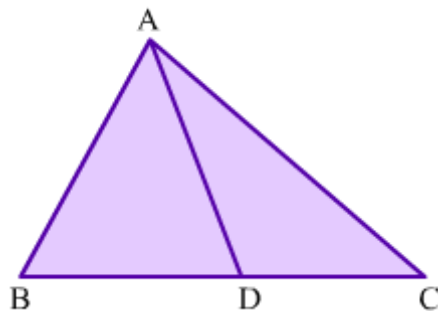
Clearly,  $\angle EFD$  has the smallest measure in the given triangle. So, the side opposite it, i.e., DE is the shortest side of the triangle.

**Medium**

**Example 1:** In the given  $\triangle ABC$ ,  $\angle ABD > \angle ACD$  and AD is the bisector of  $\angle BAC$ .

Prove that:

1.  $AB < AC$
2.  $\angle ADB < \angle ADC$



**Solution:**

1. It is given that  $\angle ABD > \angle ACD$ .

$\therefore AC > AB$  ( $\because$  Side opposite greater angle is longer)

Or,  $AB < AC$

2. In  $\triangle ABD$ , we have:

$\angle ABD + \angle BAD + \angle ADB = 180^\circ$  (By the angle sum property)

$\Rightarrow \angle ACD + \angle BAD + \angle ADB < 180^\circ$  ( $\because \angle ABD > \angle ACD$ )

$\Rightarrow \angle ACD + \angle CAD + \angle ADB < 180^\circ$  ( $\because$  AD bisects  $\angle BAC$ ;  $\angle BAD = \angle CAD$ )

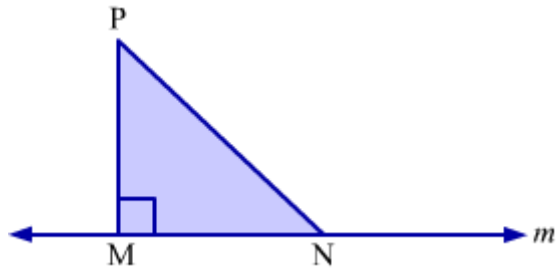
$\Rightarrow \angle ADB < 180^\circ - (\angle ACD + \angle CAD)$

$\Rightarrow \angle ADB < \angle ADC$  (By the angle sum property in  $\triangle ADC$ )

**Example 2:** Prove that of all the line segments that can be drawn to a given line from a point lying outside the line, the perpendicular line segment is the shortest.

**Solution:**

Let there be a straight line  $m$ , a point P lying outside the line and a point M lying on the line. Also,  $PM \perp m$  and N is any point other than M on  $m$ .



Since PM is perpendicular to  $m$ ,  $\angle PMN = 90^\circ$ .

In  $\triangle PMN$ , we have:

$\Rightarrow \angle PMN + \angle MPN + \angle PNM = 180^\circ$  (By the angle sum property)

$\Rightarrow 90^\circ + \angle MPN + \angle PNM = 180^\circ$

$\Rightarrow \angle MPN + \angle PNM = 90^\circ$

$\Rightarrow \angle PNM < 90^\circ$

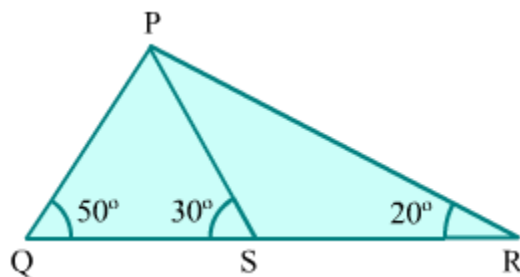
$\Rightarrow \angle PNM < \angle PMN$

$\Rightarrow PM < PN$  ( $\because$  Side opposite greater angle is longer)

Thus, PM is the shortest of all line segments from point P to line  $m$ .

### Hard

**Example 1:** In the given  $\triangle PQR$ ,  $\angle PQR = 50^\circ$ ,  $\angle PRQ = 20^\circ$  and  $\angle PSQ = 30^\circ$ . Prove that  $QS > SR$ .



**Solution:**

In  $\triangle PQS$ , we have:

$\angle PQR = 50^\circ$  and  $\angle PSQ = 30^\circ$  (Given)

By using the angle sum property in  $\triangle PQS$ , we obtain:

$$\angle PQR + \angle PSQ + \angle QPS = 180^\circ$$

$$\Rightarrow 50^\circ + 30^\circ + \angle QPS = 180^\circ$$

$$\Rightarrow \angle QPS = 180^\circ - (50^\circ + 30^\circ)$$

$$\Rightarrow \angle QPS = 100^\circ$$

So,  $\angle QPS > \angle PQR > \angle PSQ$

Since the side opposite the greater angle is longer, we get:

$$QS > PS > PQ \dots (1)$$

$\angle PSQ$  and  $\angle PSR$  form a linear pair.

$$\text{So, } \angle PSQ + \angle PSR = 180^\circ$$

$$\Rightarrow 30^\circ + \angle PSR = 180^\circ$$

$$\Rightarrow \angle PSR = 180^\circ - 30^\circ$$

$$\Rightarrow \angle PSR = 150^\circ$$

In  $\triangle PSR$ , we have:

$$\angle PSR + \angle SPR + \angle PRQ = 180^\circ \text{ (By the angle sum property)}$$

$$\Rightarrow 150^\circ + \angle SPR + 20^\circ = 180^\circ$$

$$\Rightarrow \angle SPR = 180^\circ - (150^\circ + 20^\circ)$$

$$\Rightarrow \angle SPR = 10^\circ$$

So,  $\angle PSR > \angle PRQ > \angle SPR$

Since the side opposite the greater angle is longer, we get:

$$PR > PS > SR \dots (2)$$

By using (1) and (2), we can conclude that:

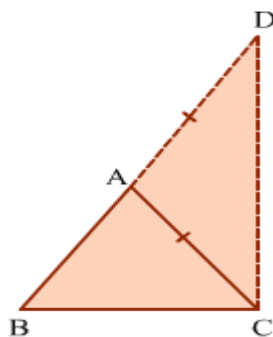
$$QS > SR$$

**Example 2:** Prove that the difference between any two sides of a triangle is less than the third side.

**Solution:**

We have to prove that the difference between any two sides of a triangle (let us say  $\triangle ABC$ ) is less than the third side of the triangle. We will do so by showing that in  $\triangle ABC$ ,  $BC - AC < AB$ .

**Construction:** Extend side BA up to point D such that  $AD = AC$ . Join C to D.



In  $\triangle ACD$ , we have:

$$AD = AC \dots (1)$$

We know that in an isosceles triangle, the angles opposite equal sides are equal.

$$\therefore \angle ACD = \angle ADC$$

$$\Rightarrow \angle ACD + \angle ACB > \angle ADC$$

$$\Rightarrow \angle BCD > \angle ADC$$

We know that the side opposite the greater angle is longer. So, we obtain,  $BD > BC$

$$\Rightarrow AB + AD > BC \quad (\because BD = AB + AD)$$

$$\Rightarrow AB + AC > BC \quad (\text{Using equation 1})$$

$$\Rightarrow BC - AC < AB$$

Similarly, we can prove that  $AB - BC < AC$  and  $AC - AB < BC$ .

Thus, we have proved that the difference between any two sides of a triangle is less than the third side.